On the mean field approximation of many-boson dynamics.

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September 13, 2016

Abstract

We show under general assumptions that the mean-field approximation for quantum many-boson systems is correct. Our contribution unifies and improves on most of the known results. The proof uses general properties of quantization in infinite dimensional spaces, phase-space analysis and measure transportation techniques.

Keywords: mean-field limit, second quantization, Wigner measures, continuity equation // 2010 Mathematics subject classification: 81S05, 81T10, 35Q55, 28A33

1 Introduction

The mean field theory for many-body quantum systems is an extensively studied mathematical subject (see for instance [1, 11, 12, 16, 18, 19, 20, 22, 24, 28, 29] and [23, 26, 41] for more old results). The main addressed question in this field is the accuracy of the mean-field approximation. While this problem is now well-understood for the most significant examples of quantum mechanics, it has no satisfactory general answer. The reason is that all the known results are concerned either with a specific model or a specific choice of quantum states. Our aim here is to show that the mean field approximation for bosonic systems is rather a general principle that depends very little on these above-mentioned specifications. The Hamiltonian of many-boson systems have formally the following form

$$H_N = \sum_{i=1}^{N} A_i + \frac{1}{N} \sum_{1\leqslant i<j\leqslant N} q_{i,j}^{(N)} = H_N^0 + q_N \,,$$

where A is a one particle kinetic energy and $q_{i,j}^{(N)}$ is a pair interaction potential between the i^{th} and j^{th} particles. It could be significant to include multi-particles interactions but to keep the presentation as simple as possible we avoid to so (see [8, 33, 16]). Assume that H_N is a self-adjoint operator on some symmetric tensor product space $\bigvee^N \mathcal{Z}_0$. Then according to the Heisenberg equation the quantum dynamics yield the time-evolved states,

$$\varrho_N(t) := |e^{-itH_N}\Psi^{(N)}\rangle\langle e^{-itH_N}\Psi^{(N)}| \,.$$

The mean-field approximation provides the first asymptotics of physical measurements in the state $\varrho_N(t)$ when the number of particles N is large. Precisely, the approximation deals with the following quantities,

$$\lim_{N\to\infty} \mathrm{Tr}[\varrho_N(t)\, B\otimes 1^{\otimes(N-k)}] \,,$$

where B is a given observable on the k first particles. Actually one can prove that, up to extracting a subsequence, there exists a Borel probability measure μ_0 on $\mathcal{Z}_0$ such that

$$\lim_{N\to\infty} \mathrm{Tr}[\varrho_N(0)\, B\otimes 1^{\otimes(N-k)}] = \int_{\mathcal{Z}_0} \langle z^{\otimes k}, Bz^{\otimes k}\rangle_{\vee^k \mathcal{Z}_0}\, d\mu_0(z) \,, \tag{1}$$

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for any compact operator $B \in \mathcal{L}(\bigvee^k \mathcal{Z}_0)$, $1 \leq k \leq N$ (k is kept fixed while $N \to \infty$). Such a result is proved in [8] and it is related to a De Finetti quantum theorem [15, 32]. So this allows to understand the structure of the above limit (1) at time $t = 0$ and there is indeed no loss of generality if we suppose that (1) holds true for the sequence of states $(\varrho_N(0))_{N\in\mathbb{N}}$. Once this is observed then the mean-field approximation precisely says that for all times $t \in \mathbb{R}$,

$$\lim_{N\to\infty} \mathrm{Tr}[\varrho_N(t)\, B \otimes 1^{\otimes(N-k)}] = \int_{\mathcal{Z}_0} \langle z^{\otimes k}, B z^{\otimes k}\rangle_{\vee^k \mathcal{Z}_0}\, d\mu_t(z)\,, \tag{2}$$

where $\mu_t = \Phi(t,0)_*\mu_0$ is a push-forward measure, μ_0 is given by (1) and $\Phi(t,0)$ is the nonlinear flow which solves the mean-field classical equation on $\mathcal{Z}_0$,

$$i\partial_t z = Az + F(z)\,. \tag{3}$$

Here the nonlinear term $F(z)$ is related to the interaction q_N and the equation (3) provides the mean-field dynamics (for instance Hartree or NLS type equations). In this article we prove the statement (2) within an abstract framework and under general assumptions. It is common to express the mean-field limit with the language of reduced density matrices. So, we remark that (2) implies the convergence of reduced density matrices in the trace-class norm (see [8] for a proof of this fact). There are essentially two requirements for the accuracy of the mean field approximation. The first concerns the regularity of the states $\varrho_N(0)$ and the second deals with the criticality of the interaction q_N. So, we assume that the quantum states have asymptotically finite kinetic energy at time $t = 0$, i.e.:

$$\mathrm{Tr}[\varrho_N(0)\, A \otimes 1^{\otimes(N-1)}] \leq C\,, \tag{4}$$

uniformly in N (A is the one particle kinetic energy). This is a reasonable requirement and in some sense a minimal one if we use energy type methods to deal with the quantum and classical dynamics. We give here only some informal insight on the assumption on q_N. Our main result is presented in detail in the next section and it is based on the abstract conditions (**D1**)-(**D2**). Suppose that A is the fractional Laplacian $(-\Delta)^s$, $s > 0$, in $L^2(\mathbb{R}^d)$ and the interaction q_N is given by

$$q_N := \frac{1}{N} \sum_{1\leqslant i<j\leqslant N} W(x_i - x_j)\,, \quad x_i, x_j \in \mathbb{R}^d\,, \tag{5}$$

where $W : \mathbb{R}^d \to \mathbb{R}$ is a measurable function. Roughly speaking, our result says that the mean-field approximation holds true in general for states satisfying (4) if:

- The system is confined and the interaction W is subcritical.
- The interaction W is subcritical with some decay at infinity.
- The system is confined and the interaction W is critical.

If the system is not confined and the interaction W is critical then we do not expect the mean-field approximation to be true for all states with the regularity (4). However, if we are able to prove higher regularity on the quantum states $\varrho_N(t)$ for all times then it is possible to justify the mean-field limit as in the other cases. Here subcritical/critical means that the interaction W belongs to $L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ with subcritical/critical exponent with respect to the kinetic energy $A = (-\Delta)^s$ according to the Sobolev embedding $H^{\frac{s}{2}}(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$. This emphasizes in particular the fact that the accuracy of the mean-field approximation depends very much on the criticality of the interaction and the regularity of initial states rather than the structure of the initial states or the exact model considered.

The method we use follows the one introduced in [9] which is based on general properties of Wick quantization in infinite dimensional spaces, Wigner measures and measure transportation techniques. We improve and simplify this method at several steps. For instance we consider only states $\varrho_N(0)$ in the symmetric tensor product $\bigvee^N \mathcal{Z}_0$ and avoid to work with states in the symmetric Fock space. This simplifies and strengthes the intermediate results. Moreover, the key argument related to convergence is clarified (see Section 5). The adaptation of measure transportation techniques in [3] to non-homogenous PDE was done in [9] with a somewhat strong condition on a related velocity field (see the assumption (**C1**) compared to the one used in [9]). This restricted the type of nonlinearity $F(z)$ that can be handled with this method. An improvement to a wider setting, briefly presented in Appendix B, is achieved in detail in [5].

As an illustration of the Wigner measures techniques used in this article, we also recover a result proved in [32] concerning the limit of the ground state energy of H_N when the system of bosons is trapped.

Overview: In the following section our main result is presented in detail and illustrated with several examples. Self-adjointness and existence of the quantum dynamics is discussed in Section 3. The proof of our main Theorem 2.3 goes through three steps: A Duhamel's formula in Section 4, a convergence argument in Section 5 and a uniqueness result for a Liouville equation in Section 6. The technical tools used along the article are explained in Appendix A and B and concern the Wick quantization, Wigner measures and transport along characteristics curves.

2 Preliminaries and results

In this section we introduce a general abstract setting suitable for the study of Hamiltonians of many-boson systems. Then we briefly recall the notion of Wigner measures and state the main results of the present article. We will often use conventional notations. In particular, the Banach space of bounded (resp. compact) operators from one Hilbert space $\mathfrak{h}_1$ into another one $\mathfrak{h}_2$ is denoted by $\mathcal{L}(\mathfrak{h}_1, \mathfrak{h}_2)$ (resp. $\mathcal{L}^\infty(\mathfrak{h}_1, \mathfrak{h}_2)$). If C (resp. q) is an operator (resp. a quadratic form) on a Hilbert space then $D(C)$ (resp. $Q(q)$) denotes its domain. In particular, if C is a self-adjoint operator then $Q(C)$ denotes its form domain (i.e. the subspace $D(|C|^{\frac{1}{2}})$).

General framework: Let $\mathcal{Z}_0$ be a separable Hilbert space. The n-fold tensor product of $\mathcal{Z}_0$ is denoted by $\otimes^n \mathcal{Z}_0$. There is a canonical action $\sigma \in \Sigma_n \to \Pi_\sigma$ of the $n-$th symmetric group Σ_n on $\otimes^n \mathcal{Z}_0$ verifying

$$\Pi_\sigma f_1 \otimes ... \otimes f_n = f_{\sigma_1} \otimes \cdots \otimes f_{\sigma_n} \,. \tag{6}$$

Hence each Π_σ extends to an unitary operator on $\otimes^n \mathcal{Z}_0$ with the relation $\Pi_\sigma \Pi_{\sigma'} = \Pi_{\sigma\circ\sigma'}$ satisfied for any $\sigma, \sigma' \in \Sigma_n$. Furthermore, the average of all these operators $(\Pi_\sigma)_{\sigma\in\Sigma_n}$, i.e.

$$\mathcal{S}_n = \frac{1}{n!} \sum_{\sigma\in\Sigma_n} \Pi_\sigma \,, \tag{7}$$

defines an orthogonal projection on $\otimes^n \mathcal{Z}_0$. By definition the symmetric n-fold tensor product of $\mathcal{Z}_0$ is the Hilbert subspace

$$\bigvee^n \mathcal{Z}_0 = \mathcal{S}_n(\mathcal{Z}_0^{\otimes n}) \,.$$

Consider now an operator A on $\mathcal{Z}_0$ and assume that:

Assumption (A1):

$$A \textit{ is a non-negative and self-adjoint operator on } \mathcal{Z}_0. \tag{\textbf{A1}}$$

For $i = 1, \cdots, n$, let

$$A_i = 1^{\otimes(i-1)} \otimes A \otimes 1^{\otimes(n-i)},$$

where the operator A in the right hand side acts on the i^{th} component. The free Hamiltonian of a many-boson system is

$$H_N^0 = \sum_{i=1}^N A_i \,, \tag{8}$$

which is a self-adjoint non-negative operator on $\bigvee^N \mathcal{Z}_0$. In order to introduce a two particles interaction in an abstract setting we consider a symmetric quadratic form q on $Q(A_1 + A_2) \subset \otimes^2 \mathcal{Z}_0$. Here $A_1 + A_2$ is considered as an operator on $\otimes^2 \mathcal{Z}_0$ and the subspace $Q(A_1 + A_2)$ contains non-symmetric vectors. Throughout this paper we assume:

Assumption (A2):

$$\begin{aligned} &q \textit{ is a symmetric sesquilinear form on } Q(A_1+A_2) \textit{ satisfying :}\\ &\exists\, 0 < a < 1,\, b > 0,\ \ \forall u \in Q(A_1 + A_2),\ |q(u, u)| \leq a\langle u, (A_1 + A_2)u\rangle + b\|u\|^2_{\otimes^2\mathcal{Z}_0} \,. \end{aligned} \tag{\textbf{A2}}$$

As a consequence of the above assumption, q can be identified with a bounded operator $\tilde{q}$, satisfying the relation:

$$q(u, v) = \langle u, \tilde{q}\, v\rangle_{\otimes^2\mathcal{Z}_0} \,, \quad \forall u, v \in Q(A_1 + A_2) \,, \tag{9}$$

and $\tilde{q}$ acts from the Hilbert space $Q(A_1 + A_2)$ equipped with the graph norm into its dual $Q'(A_1 + A_2)$ with respect to the inner product of $\otimes^2 \mathcal{Z}_0$.
Now, we define a collection of quadratic forms $(q_{i,j}^{(n)})_{1\leq i<j\leq n}$ by

$$q_{i,j}^{(n)}(\varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_n, \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n) = q(\varphi_i \otimes \varphi_j, \psi_i \otimes \psi_j) \prod_{k\neq i,j} \langle \varphi_k, \psi_k \rangle\,, \tag{10}$$

for any $\varphi_1, \cdots, \varphi_n, \psi_1, \cdots, \psi_n$ in $Q(A)$. By linearity all the $q_{i,j}^{(n)}$ extend to well defined quadratic forms on the algebraic tensor product $\otimes^{alg,n} Q(A)$. Using the assumptions (**A1**)-(**A2**), we prove in Lemma 3.1 that each $q_{i,j}^{(n)}$, $1 \leq i < j \leq n$, extends uniquely to a symmetric quadratic form on $Q(H_n^0) \subset \bigvee^n \mathcal{Z}_0$.
We now consider the *many-boson Hamiltonian* to be the quadratic form on $Q(H_N^0)$ given by

$$H_N = \sum_{i=1}^{N} A_i + \frac{1}{N} \sum_{1\leqslant i<j\leqslant N} q_{i,j}^{(N)} = H_N^0 + q_N. \tag{11}$$

Actually the assumptions (**A1**)-(**A2**) imply the existence of the many-boson dynamics, since there exists a unique self-adjoint operator, denote again by H_N, associated to the quadratic form (11) (see Proposition 3.4).

The classical dynamics: Let $(Q(A), \|\cdot\|_{Q(A)})$ be the domain form of the non-negative self-adjoint operator A equipped with the graph norm,

$$\|u\|_{Q(A)}^2 = \langle u, (A+1)u \rangle,\ u \in Q(A)\,,$$

and $Q'(A)$ its dual with respect to the inner product of $\mathcal{Z}_0$. The quadratic form q defines a quartic monomial,

$$z \in Q(A) \mapsto q_0(z) := \frac{1}{2} q(z^{\otimes 2}, z^{\otimes 2})\,,$$

which is Gâteaux differentiable on $Q(A)$. Hence one can define the Gâteaux derivative of q_0 with respect to $\bar{z}$ according to the formula:

$$\partial_{\bar{z}} q_0(z)[u] = \frac{1}{2} \partial_{\bar{\lambda}} q((z+\lambda u)^{\otimes 2}, (z+\lambda u)^{\otimes 2})_{|\bar{\lambda}=0}, \tag{12}$$

where $\partial_{\bar{\lambda}}$ is the Wirtinger derivative in the complex field $\mathbb{C}$. For each $z \in Q(A)$, the map $u \mapsto \partial_{\bar{z}} q_0(z)[u]$ is a anti-linear continuous form on $Q(A)$ and hence $\partial_{\bar{z}} q(z)$ can be identified with a vector $\partial_{\bar{z}} q_0(z) \in Q'(A)$ by the Riesz representation theorem. In the sequel, we set $v_t(z) := -ie^{itA} \partial_{\bar{z}} q_0(e^{-itA} z) : \mathbb{R} \times Q(A) \to \mathcal{Z}_0$. Assume that the velocity field satisfies the following assumption.

Assumption (C1):

For any $t \in \mathbb{R}$, and $M > 0$, there exists $C(M) > 0$ such that:

$$||v_t(z) - v_t(y)||_{\mathcal{Z}_0} \leq C(M) \left(||z||_{Q(A)}^2 + ||y||_{Q(A)}^2\right) ||z-y||_{\mathcal{Z}_0}\,, \tag{\textbf{C1}}$$

for all $z, y \in Q(A)$ such that $||z||_{\mathcal{Z}_0}, ||y||_{\mathcal{Z}_0} \leq M$.

We shall assume that the classical mean field equation in the interaction representation.

$$\begin{cases} \dot{\gamma}(t) = v_t(\gamma(t))\,, \\ \\ \gamma(s) = z \in Q(A)\,, \quad s \in I\,, \end{cases} \tag{13}$$

is locally well posed in $Q(A)$. We say that a solution $I \ni t \mapsto \gamma(t)$ of the Cauchy problem is strong if it belongs to the space $\mathcal{C}(I, Q(A)) \cap \mathcal{C}^1(I, Q'(A))$. We recall that the Cauchy problem (3) and (13) are equivalent, i.e. z is a strong solution of (3) if and only if $\tilde{z} := e^{itA} z$ is a strong solution of (13).

Assumption (C2):

The Cauchy problem (13) *is (LWP) in $Q(A)$:*

i) There exists a open ball B of $\mathscr{Z}_0$ such that for any $z_0 \in B \cap Q(A)$, the Borel flow

$$\Phi(t,0) : B \cap Q(A) \to B \cap Q(A),$$

is well defined for any $t \in \bar{I}$, and $z_t = \Phi(t,0)z_0$ is a strong solution of the Cauchy problem (13).

(ii) Continuous dependence on initial data: If $z_n \to z$ in $Q(A)$ and $J \subset (T_{min}(z,s), T_{max}(z,s))$ is a closed interval, then for n large enough the strong solutions γ_n of (13) *provided by (ii) with $\gamma_n(s) = z_n$ are defined on J and satisfy $\gamma_n \underset{n\to\infty}{\to} \gamma$ in $C(J, Q(A))$.*

If $I = \mathbb{R}$ in (i) for any $z \in Q(A)$ and any $s \in \mathbb{R}$, we say that the initial value problem is globally well-posed (GWP).

Remarks 2.1. *Assumption* (**C1**) *implies the uniqueness of solution in $L^\infty(I, Q(A)) \cap W^{1,\infty}(I, Q'(A))$, where $W^{1,p}(I, Q'(A))$, for $1 \le p \le \infty$, denote the Sobolev spaces of classes of functions in $L^p(I, Q'(A))$ with distributional first derivatives in $L^p(I, Q'(A))$.*

The Wigner measures: The mean-field problem is tackled here through the Wigner measures method elaborated in [7, 8]. The idea of these measures has its roots in the finite dimensional semi-classical analysis. It allows to generalize the notion of mean-field convergence to states that are not coherent nor factorized. For ease of reading, we briefly recall their definition here while their main features are discussed in Appendix A.

Definition 2.2. *Let $\{\varrho_N := |\Psi^{(N)}\rangle\langle\Psi^{(N)}|\}_{N\in\mathbb{N}}$ be a sequence of normal states on $\vee^N \mathscr{Z}_0$, i.e $\|\Psi^{(N)}\|_{\vee^N \mathscr{Z}_0} = 1$. The set $\mathcal{M}(\varrho_N, N \in \mathbb{N})$ of Wigner measures of $(\varrho_N)_{N\in\mathbb{N}}$ is the set of Borel probability measures on $\mathscr{Z}_0$, μ , such that there exists a subsequence $(N_k)_{k\in\mathbb{N}}$ satisfying:*

$$\forall \xi \in \mathscr{Z}_0 \,, \quad \lim_{k\to+\infty} \langle \Psi^{(N_k)}, \mathcal{W}(\sqrt{2}\pi\xi)\Psi^{(N_k)}\rangle = \int_{\mathscr{Z}_0} e^{2i\pi \mathrm{Re}\langle \xi, z\rangle} \, d\mu(z) \,, \tag{14}$$

where $\mathcal{W}(\sqrt{2}\pi\xi)$ is the Weyl operator in the symmetric Fock space defined in (60) *with $\varepsilon = \frac{1}{N_k}$.*

The right hand side of (14) is the inverse Fourier transform of the measure μ. So Wigner measures are identified through their characteristic functions. Moreover, it was proved in [6, Theorem 6.2] that the set $\mathcal{M}(\varrho_N, N \in \mathbb{N})$ is non-empty and according to [6, 8, 9] it is a convenient tool for the study the mean-field limit. In particular, it allows to understood the convergence of reduced density matrices (2), which are the main analyzed quantities in other approaches ([41]).

2.1 Results

Dynamical result: Our main result concerns the effectiveness of the mean field approximation for general N-particle states and under general assumptions (**D1**)-(**D2**). We prove that the time-dependant Wigner measures of evolved states $\varrho_N(t) := |e^{-itH_N}\Psi^{(N)}\rangle\langle e^{-itH_N}\Psi^{(N)}|$ are the push-forward of the initial measures (associated with the initial states $\varrho_N(0)$) by the global flow of the field equation. Eventually, if $\varrho_N(0)$ has only one Wigner measure then $\varrho_N(t)$ will have also one single Wigner measure described as above. Moreover, the result is applicable to either trapped or untrapped systems of bosons.

Assumption (D1):

A has compact resolvent and there exists a subspace D dense in $Q(A)$ such that for any $\xi \in D$,

$$\begin{aligned}
&\lim_{\lambda\to+\infty} ||\langle\xi| \otimes (A+1)^{-\frac{1}{2}}\, \mathcal{S}_2\, \tilde{q}(A_1 + A_2 + \lambda)^{-\frac{1}{2}}||_{\mathcal{L}(\bigvee^2 \mathscr{Z}_0, \mathscr{Z}_0)} = 0 \,, \\
&\lim_{\lambda\to+\infty} ||\langle\xi| \otimes (A+\lambda)^{-\frac{1}{2}}\, \mathcal{S}_2\, \tilde{q}(A_1 + A_2 + 1)^{-\frac{1}{2}}||_{\mathcal{L}(\bigvee^2 \mathscr{Z}_0, \mathscr{Z}_0)} = 0 \,.
\end{aligned} \tag{D1}$$

Actually, by Assumption (**A2**), the operator $(A_1+A_2+1)^{-\frac{1}{2}}\tilde{q}(A_1+A_2+1)^{-\frac{1}{2}}$ is bounded but usually not compact in applications. Our second main assumption is given below and it implies the two limits in (**D1**).

Assumption (D2):

There exists a subspace D dense in $Q(A)$ such that for any $\xi \in D$,

$$\langle \xi | \otimes (A+1)^{-\frac{1}{2}} \mathcal{S}_2 \, \tilde{q}(A_1 + A_2 + 1)^{-\frac{1}{2}} \in \mathcal{L}^\infty(\bigvee^2 \mathcal{Z}_0, \mathcal{Z}_0) . \qquad \textbf{(D2)}$$

The Assumption concerns the symbol derivative $\partial_{\bar z} q$ which is supposed to be a compact form-perturbation of the free Hamiltonian $(A_1 + A_2)$ or A. Consider the abstract setting explained above with a separable Hilbert space $\mathcal{Z}_0$, a one-particle self-adjoint operator A and a two-body interaction q. Then our main result on the dynamical mean-field problem is stated below.

Theorem 2.3. *Assume* (**A1**)-(**A2**)-(**C1**)-(**C2**) *and suppose that either* (**D1**) *or* (**D2**) *holds true. Let $\{\varrho_N = |\Psi^{(N)}\rangle\langle\Psi^{(N)}|\}_{N\in\mathbb{N}}$ a sequence of normal states on $\bigvee^N \mathcal{Z}_0$ with a unique Wigner measure μ_0 and satisfying:*

$$\exists C > 0, \forall N \in \mathbb{N}, \;\; \langle \Psi^{(N)}, H_N^0 \Psi^{(N)} \rangle \leq CN . \tag{15}$$

Then for any time $t \in \mathbb{R}$, the family $\{\varrho_N(t) = |e^{-itH_N}\Psi^{(N)}\rangle\langle e^{-itH_N}\Psi^{(N)}|\}_{N\in\mathbb{N}}$ has a unique Wigner measure μ_t which is a Borel probability measure on $Q(A)$. In addition, for any time $t \in \mathbb{R}$, $\mu_t = \Phi(t,0)_*\mu_0$, the push-forward of the initial measure μ_0 by the globally well-defined flow $\Phi(t,0)$ associated to the field equation:

$$\begin{cases} i\partial_t z = Az + \partial_{\bar z} q_0(z) \\ z_{|t=0} = z_0. \end{cases} \tag{16}$$

Remarks 2.4. *1) The above theorem remains true if we assume that A is semi-bounded from below.*
2) It is not necessary to assume that ϱ_N admits a unique Wigner measure μ_0. In general the result says:

$$\mathcal{M}(\varrho_N(t), N \in \mathbb{N}) = \{\Phi(t,0)_*\mu_0, \mu_0 \in \mathcal{M}(\varrho_N, N \in \mathbb{N})\} .$$

3) Without essential changes in the proof of Theorem 2.3, we can suppose that ϱ_N is an arbitrary sequence of non-negative trace-class operator on $\bigvee^N \mathcal{Z}_0$ satisfying:

$$\exists C > 0, \forall N \in \mathbb{N}, \;\; \mathrm{Tr}[\varrho_N H_N^0] \leq CN, \text{ and } \exists C' > 0, \mathrm{Tr}[\varrho_N] \leq C'.$$

4) The estimate (15) *implies that the Wigner measure μ_t associated with the family $\varrho_N(t)$ is carried on $Q(A)$ by Proposition A.6 and Proposition 3.5. Moreover, the family of normal states $\varrho_N(t)$ is satisfying*

$$\langle e^{itH_N}\Psi^{(N)}, \mathbf{N} e^{itH_N}\Psi^{(N)} \rangle = 1,$$

where $\mathbf{N}$ is the Number operator defined in (61) *and subsequently the Wigner measure μ_t is also carried on the unit ball of $\mathcal{Z}_0$.*

Variational result: The second result concerns the ground state energy of trapped many-boson systems in the mean-field limit. It follows directly from the key Lemma 7.1. This result is already proved in a general framework in [32] using a quantum De Finetti theorem. Consider the Hamiltonian H_N given by (11) and suppose that (**A1**)-(**A2**) are satisfied. The confinement of the system is equivalent to the requirement that the operator A has compact resolvent. By definition the quantum ground state energy is

$$E(N) := \inf_{\substack{\Psi^{(N)} \in Q(H_N^0) \\ \|\Psi^{(N)}\|_{\bigvee^N \mathcal{Z}_0} = 1}} \langle \Psi^{(N)}, H_N \Psi^{(N)} \rangle .$$

On the other hand the classical energy functional is

$$h(z) = \langle z, Az \rangle + \frac{1}{2} q(z^{\otimes 2}, z^{\otimes 2}), \quad \forall z \in Q(A) . \tag{17}$$

Using (**A2**), one observes that $\inf_{z\in Q(A), \|z\|_{\mathcal{Z}_0}=1} h(z)$ is finite. In fact for any $z \in Q(A)$ such that $\|z\|_{\mathcal{Z}_0} = 1$,

$$h(z) \geq (1-a)\langle z^{\otimes 2}, (A_1 + A_2) z^{\otimes 2} \rangle - C_1 \geq -C_1.$$

Theorem 2.5. *Assume* (**A1**)-(**A2**) *and suppose that A has compact resolvent. Then*

$$\lim_{N\to+\infty} \frac{E(N)}{N} = \lim_{N\to+\infty} \frac{1}{N} \inf_{\substack{\Psi^{(N)} \in Q(H_N^0) \\ \|\Psi^{(N)}\| = 1}} \langle \Psi^{(N)}, H_N \Psi^{(N)} \rangle = \inf_{\substack{z\in Q(A) \\ \|z\|_{\mathcal{Z}_0}=1}} h(z) > -\infty.$$

2.2 Examples

In this section, we provide several examples to which the general result of Theorem 2.3 is applicable. But first we observe that the two limits in (**D1**) are satisfied whenever q is infinitesimally A_1+A_2-form bounded. This indeed allows to handle the situation when the interaction is "subcritical". But when the interaction is comparable to the kinetic energy we rely directly on (**D1**) which seems to be the appropriate assumption in this case.

Lemma 2.6. *Assume* (**A1**)-(**A2**) *and suppose that the quadratic form q is infinitesimally A_1+A_2-form bounded. Then for any $\xi \in Q(A)$,*

$$\lim_{\lambda\to+\infty} ||\langle\xi| \otimes (A+1)^{-\frac{1}{2}}\, \mathcal{S}_2\, \tilde{q}(A_1+A_2+\lambda)^{-\frac{1}{2}}||_{\mathcal{L}(\bigvee^2 \mathcal{Z}_0,\mathcal{Z}_0)} = 0\,,$$

$$\lim_{\lambda\to+\infty} ||\langle\xi| \otimes (A+\lambda)^{-\frac{1}{2}}\, \mathcal{S}_2\, \tilde{q}(A_1+A_2+1)^{-\frac{1}{2}}||_{\mathcal{L}(\bigvee^2 \mathcal{Z}_0,\mathcal{Z}_0)} = 0\,.$$

Proof. Let $\Phi \in \mathcal{Z}_0$ and $\Psi \in \bigvee^2 \mathcal{Z}_0$ then by Cauchy-Schwarz inequality,

$$\begin{aligned} |\langle\Phi, \langle\xi| \otimes (A+1)^{-\frac{1}{2}}\, \mathcal{S}_2\, \tilde{q}(A_1+A_2+\lambda)^{-\frac{1}{2}}\,\Psi\rangle| &= |q(\mathcal{S}_2\xi\otimes(A+1)^{-\frac{1}{2}}\Phi, (A_1+A_2+\lambda)^{-\frac{1}{2}}\,\Psi)| \\ &\leq |q(\mathcal{S}_2\xi\otimes(A+1)^{-\frac{1}{2}}\Phi)|^{\frac{1}{2}}\;\; |q((A_1+A_2+\lambda)^{-\frac{1}{2}}\,\Psi)|^{\frac{1}{2}}\,, \end{aligned}$$

with $q(u)=q(u,u)$. Remark that $|q(\mathcal{S}_2\xi\otimes(A+1)^{-\frac{1}{2}}\Phi)|$ is bounded thanks to (**A2**) and the fact that $\xi\in Q(A)$. Since q is infinitesimally A_1+A_2-form bounded, then for any $\alpha>0$ there exists $C(\alpha)>0$ such that

$$\begin{aligned} |q((A_1+A_2+\lambda)^{-\frac{1}{2}}\,\Psi)| &\leq \alpha\langle\Psi, (A_1+A_2+\lambda)^{-1}(A_1+A_2+\frac{C(\alpha)}{\alpha})\,\Psi\rangle \\ &\leq \max(\alpha, \frac{C(\alpha)}{\lambda})\,\|\Psi\|\,. \end{aligned}$$

This proves the first limit in (**D1**) when $\lambda\to\infty$. The second one follows by a similar argument. □

Example 1 (The two-body delta interaction)**.** *Non-relativistic systems of trapped bosons with a two-body point interaction,*

$$H_N = \sum_{i=1}^{N} -\Delta_{x_i} + V(x_i) + \frac{\kappa}{N}\sum_{1\leqslant i<j\leqslant N}\delta(x_i-x_j), \qquad x_i, x_j\in\mathbb{R},\ \kappa\in\mathbb{R}, \tag{18}$$

where δ is the Dirac distribution and V is a real-valued potential which splits into two parts $V=V_1+V_2$ such that

$$V_1\in L^1_{loc}(\mathbb{R}),\ V_1\geq 0\,,\ \lim_{|x|\to+\infty} V_1(x) = +\infty\,,$$

$$V_2 \textit{ is } -\Delta\textit{-form bounded with a relative bound less than one.}$$

This model has been studied for instance in [1, 4]. The operator $A=-\Delta+V$ is self-adjoint semi-bounded from below and A has compact resolvent according to [39, Theorem X19]. The two-body interaction q is given by $q(z^{\otimes 2}, z^{\otimes 2}) = \kappa\langle z^{\otimes 2}, \delta(x_1-x_2)z^{\otimes 2}\rangle = \kappa\|z\|^4_{L^4(\mathbb{R})}$ and satisfies for any $u\in Q(A_1+A_2)$,

$$\forall\alpha>0,\ |q(u,u)| \leq \frac{\alpha\kappa}{2\sqrt{2}}\langle u, A_1+A_2\, u\rangle + \frac{\kappa}{4\alpha\sqrt{2}}\|u\|^2_{L^2(\mathbb{R}^2)}.$$

For a detailed proof of the latter inequality see [4, Lemma A.1]. Hence (**A1**)-(**A2**) *are verified and by Lemma 2.6 the assumption* (**D1**) *holds true. The vector field is given by $\partial_{\bar z}q_0(z) = \kappa|z|^2 z : Q(A)\to Q(A)$ and satisfies the inequalities,*

$$\forall z,y\in Q(A),\ \exists C := C(\|z\|_{Q(A)}, \|y\|_{Q(A)})>0,\ \||z|^2z-|y|^2y\|_{Q(A)} \leq C\|z-y\|_{Q(A)}\,, \tag{19}$$

and

$$\forall z,y\in Q(A),\ \||z|^2z-|y|^2y\|_{L^2(\mathbb{R})} \leq C(\|z\|^2_{H^1(\mathbb{R})}+\|y\|^2_{H^1(\mathbb{R})})\|z-y\|_{L^2(\mathbb{R})}, \tag{20}$$

since the inclusion $Q(A)\subset H^1(\mathbb{R})\subset L^\infty(\mathbb{R})$ holds by Sobolev embedding and the fact that $Q(A)=\{u\in L^2(\mathbb{R}), u'\in L^2(\mathbb{R}), V_1^{\frac{1}{2}}u\in L^2(\mathbb{R})\}$. Therefore the vector field $\partial_{\bar z}q_0(z)$ is locally Lipschitz in $Q(A)$ and the (**NLS**) *equation*

$$\begin{cases} i\partial_t z = -\Delta z + Vz + \kappa|z|^2 z \\ z_{|t=0} = z_0, \end{cases} \tag{\textbf{NLS}}$$

is locally well-posed in $Q(A)$ and assumption **(C2)** *holds true . The estimate* (20) *implies estimate* (**C1**) *for the velocity field $v_t(z) := -ie^{itA}\partial_{\bar z} q_0(e^{-itA}z)$. Furthermore, using the energy and charge conservation one shows the global well-posedness of the* (**NLS**) *equation and Theorem 2.3 holds true.*

Example 2 (Trapped bosons)**.** *Non relativistic trapped many-boson systems with singular two-body potentials:*

$$H_N = \sum_{i=1}^{N} -\Delta_{x_i} + V(x_i) + \frac{1}{N} \sum_{1\leqslant i<j\leqslant N} W(x_i - x_j), \qquad x_i, x_j \in \mathbb{R}^d. \tag{21}$$

where V is a real-valued potential which splits into two parts $V = V_1 + V_2$ such that:

$$\begin{aligned} &V_1 \in C^\infty(\mathbb{R}^d, \mathbb{R}),\ V_1 \geq 0,\ D^\alpha V_1 \in L^\infty(\mathbb{R}^d),\ \forall \alpha \in \mathbb{N}^d, |\alpha| \geq 2\,, \\ &V_1(x) \to \infty, \quad \textit{when } |x| \to \infty\,, \\ &V_2 \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d),\ p \geq 1,\ p > \frac{d}{2}\,, \end{aligned}$$

and $W : \mathbb{R}^d \to \mathbb{R}$ is an even measurable function verifying:

$$W \in L^q(\mathbb{R}^d) + L^\infty(\mathbb{R}^d),\ q \geq 1,\ q \geq \frac{d}{2}, (\textit{and } q > 1 \textit{ if } d = 2). \tag{22}$$

By Gagliardo-Nirenberg inequality we know that (22) *implies that W is infinitesimally $-\Delta$-form bounded. So, the assumptions* (**A1**)*-*(**A2**) *and* (**D1**) *are satisfied. Moreover, the vector field $[\partial_{\bar z} q_0](z) = W * |z|^2 z : Q(A) \to Q'(A)$ satisfies for any $z, y \in Q(A)$,*

$$\|W * |z|^2 z - W * |y|^2 y\|_{L^2(\mathbb{R}^d)} \lesssim \|W\|_{L^q} (\|z\|^2_{H^1(\mathbb{R}^d)} + \|y\|^2_{H^1(\mathbb{R}^d)})\, \|z - y\|_{L^2(\mathbb{R}^d)}\,. \tag{23}$$

So assumption (**C1**) *holds true. The global well-posedness in $Q(A)$, conservation of energy and charge of the Hartree equation*

$$\begin{cases} i\partial_t z = -\Delta z + Vz + W * |z|^2 z \\ z_{t=0} = z_0, \end{cases} \tag{\textbf{Hartree}}$$

are proved in [14] Theorem 9.2.6 and Remark 9.2.8, hence **(C2)** *holds true. Observe that the assumption on W are satisfied by the Coulomb type potentials $\frac{\lambda}{|x|^\alpha}$ when $\alpha < 2$, $\lambda \in \mathbb{R}$ and $d = 3$.*

Example 3 (Untrapped bosons)**.** *Non-relativistic untrapped many-boson systems,*

$$H_N = \sum_{i=1}^{N} -\Delta_{x_i} + V(x_i) + \frac{1}{N} \sum_{1\leqslant i<j\leqslant N} W(x_i - x_j), \qquad x_i, x_j \in \mathbb{R}^d.$$

where the potentials V and W satisfy the following assumptions for some p and q,

$$\begin{aligned} &V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d), p \geq 1,\ p > \frac{d}{2}, \\ &W \in L^q(\mathbb{R}^d) + L^\infty_0(\mathbb{R}^d), q \geq 1,\ q \geq \frac{d}{2}, (\textit{and } q > 1 \textit{ if } d = 2) \end{aligned} \tag{24}$$

Here $L^\infty_0(\mathbb{R}^d)$ denotes the space of bounded measurable functions going to 0 at infinity. For instance Coulomb potentials $\frac{\lambda}{|x|^\alpha}$ for $\alpha < 2$, $\lambda \in \mathbb{R}$ and $d = 3$ satisfy (24)*. As in the previous example* (**A1**)*-*(**A2**) *are satisfied and* (**D2**) *is verified if we check that $(1 - \Delta_x)^{-\frac{1}{2}} W(x) (1 - \Delta_x)^{-\frac{1}{2}}$ is compact (see the proof of [9, Lemma 3.10]). In fact W decomposes as $W = W_1 + W_2$ with $W_1 \in L^q(\mathbb{R}^d)$ and $W_2 \in L^\infty_0(\mathbb{R}^d)$. We know that $W_2(1 - \Delta)^{-\frac{1}{2}} \in \mathcal{L}^\infty(L^2(\mathbb{R}^d))$ (see for instance [27, Proposition 3.21]). Therefore we only need to check that $(1 - \Delta_x)^{-\frac{1}{2}} W_1(x) (1 - \Delta_x)^{-\frac{1}{2}}$ is compact. Let $\chi \in \mathcal{C}^\infty_0(\mathbb{R}^d)$ such that $0 \leq \chi \leq 1$ and $\chi = 1$ in a neighborhood of 0. We denote $\chi_m(x) := \chi(\frac{x}{m})$, for $x \in \mathbb{R}^d$ and $m \in \mathbb{N}^*$. For a given measurable function g let $(g^\delta)_{\delta>0}$ denotes*

$$g^\delta = \begin{cases} g, & \textit{if } |g| < \delta \\ \delta, & \textit{if } g \geq \delta \\ -\delta, & \textit{if } g \leq -\delta\,. \end{cases} \tag{25}$$

Writing the decomposition

$$W_1 = \underbrace{(\chi_m W_1)^\delta}_{L^\infty_0(\mathbb{R}^d)} + \underbrace{W_1 - (\chi_m W_1)^\delta}_{L^q(\mathbb{R}^d)},$$

we observe that

$$(1-\Delta)^{-\frac{1}{2}}(\chi_m W_1)^\delta (1-\Delta)^{-\frac{1}{2}} \in \mathcal{L}^\infty(L^2(\mathbb{R}^d)),$$

and for $\delta \to +\infty$ and $m \to +\infty$,

$$(1-\Delta)^{-\frac{1}{2}}(\chi_m W_1)^\delta (1-\Delta)^{-\frac{1}{2}} \longrightarrow (1-\Delta)^{-\frac{1}{2}} W_1 (1-\Delta)^{-\frac{1}{2}}\,, \tag{26}$$

in the norm topology. Hence (**D2**) *holds true. The convergence* (26) *is justified by the Gagliardo-Nirenberg's inequality,*

$$|\langle u, \big[(\chi_m W_1)^\delta - W_1\big] u\rangle| \le C \|(\chi_m W_1)^\delta - W_1\|_{L^q(\mathbb{R}^d)} \, \|\nabla u\|^{2\alpha}_{L^2(\mathbb{R}^d)} \, \|u\|^{2(1-\alpha)}_{L^2(\mathbb{R}^d)}\,, \qquad \alpha = \frac{d}{2q}\,.$$

As in Example 2, the vector field $-ie^{itA}\partial_{\bar z} q_0(e^{-itA}z)$ satisfies the inequality (**C1**). *The global well-posedness in $H^1(\mathbb{R}^d)$, conservation of energy and charge of the Hartree equation,*

$$\begin{cases} i\partial_t z = -\Delta z + Vz + W * |z|^2 z \\ z_{|t=0} = z_0, \end{cases}$$

hold true according to [14] Corollary 4.3.3 and Corollary 6.1.2, and assumption **(C2)** *holds true.*

Example 4 (Non-relativistic Bosons with magnetic field)**.** *Non-relativistic many-boson systems with an external magnetic field $\mathcal{A} : \mathbb{R}^d \to \mathbb{R}^d$ and an external electric field $V : \mathbb{R}^d \to \mathbb{R}$ are described by the Hamiltonian,*

$$H_N = \sum_{j=1}^N \big[(-i\nabla_{x_j} + \mathcal{A}(x_j))^2 + V(x_j)\big] + \frac{1}{N} \sum_{1 \leqslant i < j \leqslant N} W(x_i - x_j), \tag{27}$$

where $W(x)$ is an even measurable function satisfying with $\mathcal{A}$ and V the assumptions:

$$\begin{aligned}
& d \ge 3, \\
& \mathcal{A} \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}^d), \\
& V \in L^1_{loc}(\mathbb{R}^d, \mathbb{R}),\ V_+(x) \to \infty,\ \text{when } |x| \to \infty\,, \\
& V_- \text{ is } -\Delta\text{-form bounded with relative bound less than } 1, \\
& W \in L^q(\mathbb{R}^d, \mathbb{R}) + L^\infty(\mathbb{R}^d, \mathbb{R}),\ \nabla W \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d) \text{ for some } q > \frac{d}{2},\ p \ge \frac{d}{3}.
\end{aligned}$$

Here $V_\pm$ denotes the positive and negative part of the potential V. Let $\nabla_{\mathcal{A}} := \nabla + i\mathcal{A}$ then the quadratic form

$$H_V(\mathcal{A})[f,g] := \int_{\mathbb{R}^d} \overline{\nabla_{\mathcal{A}} f(x)}\, \nabla_{\mathcal{A}} g(x)\, dx + \int_{\mathbb{R}^d} V(x) \overline{f(x)} g(x)\, dx\,,$$

defined on the form domain

$$\mathcal{H}^1_{\mathcal{A},V}(\mathbb{R}^d) := \{\varphi \in L^2(\mathbb{R}^d),\ \nabla_{\mathcal{A}}\varphi,\ V_+^{\frac{1}{2}}\varphi \in L^2(\mathbb{R}^d)\},$$

is closed and bounded from below and hence it defines a unique semi-bounded from below self-adjoint operator denoted $H_V(\mathcal{A})$ (see [10],[30]). Moreover, $C_0^\infty(\mathbb{R}^d)$ is a form core for $H_V(\mathcal{A})$. Hence (**A1**) *is true and since W satisfies the condition* (22) *of Example 2 we know that $W(x_1 - x_2)$ is infinitesimally $-\Delta_{x_1} - \Delta_{x_2}$-form bounded. Applying [10, Theorem 2.5] one concludes that $W(x_1 - x_2)$ is infinitesimally $H_0(\mathcal{A}) \otimes 1 + 1 \otimes H_0(\mathcal{A})$-form bounded and subsequently it is infinitesimally $H_V(\mathcal{A}) \otimes 1 + 1 \otimes H_V(\mathcal{A})$-form bounded. Hence* (**A2**) *is also true. Moreover, according to [10, Theorem 2.7] $H_V(\mathcal{A})$ has compact resolvent and so assumption* (**D1**) *is satisfied.*
The global well-posedness in $\mathcal{H}^1_{\mathcal{A},V}(\mathbb{R}^d)$ of the Hartree equation with magnetic field

$$\begin{cases} i\partial_t z = (-i\nabla + \mathcal{A})^2 z + Vz + W * |z|^2 z \\ z_{|t=0} = z_0, \end{cases} \tag{28}$$

is proved in [35] together with energy and charge conservation and **(C2)** *holds true. Assumption* (**C1**) *holds true since by Young, Hölder and Sobolev inequalities,*

$$\|\big[W_1 * \bar z z\big] z - \big[W_1 * \bar y y\big] y\|_{L^2(\mathbb{R}^d)} \le \|W_1\|_{L^q(\mathbb{R}^d)} (\|z\|^2_{\mathcal{H}^1_{\mathcal{A},V}(\mathbb{R}^d)} + \|y\|^2_{\mathcal{H}^1_{\mathcal{A},V}(\mathbb{R}^d)}) \|z - y\|_{L^2(\mathbb{R}^d)} \tag{29}$$

Here $W = W_1 + W_2$ with $W_1 \in L^q(\mathbb{R}^d)$ and $W_2 \in L^\infty(\mathbb{R}^d)$. The mean field problem for this type of model was studied in [34].

Example 5 (Semi-relativistic bosons with critical interaction)**.** *This model has been presented in [18] or [36] to describe boson stars. Semi-relativistic systems of bosons have the many-body Hamiltonian*

$$H_N = \sum_{j=1}^{N} \sqrt{-\Delta_{x_j} + m^2} + V(x_j) + \frac{\kappa}{N} \sum_{1 \leqslant i < j \leqslant N} \frac{1}{|x_i - x_j|}, \quad x_i, x_j \in \mathbb{R}^3 ,$$

with $-\kappa_{cr} < \kappa < \kappa_{cr}$, $\kappa_{cr}^{-1} := 2 \lim_{\alpha \to \infty} || \frac{1}{|x|} (-\Delta + \alpha)^{-\frac{1}{2}} ||$, $m \geq 0$ *and* V *is real-valued measurable function* $V = V_1 + V_2$ *satisfying*

$$\begin{aligned} & V_1 \in L^1_{loc}(\mathbb{R}^3),\ V_1 \geq 0,\ V_1(x) \to \infty \text{ when } |x| \to \infty , \\ & V_2 \text{ is } \sqrt{-\Delta} - \text{form bounded with a relative bound less than } 1 . \end{aligned}$$

The quadratic form

$$\begin{aligned} & A[u,u] = \langle u, \sqrt{-\Delta + m^2}\, u \rangle + \langle u, V u \rangle , \\ & Q(A) = \{ u \in L^2(\mathbb{R}^3),\ (-\Delta + m^2)^{\frac{1}{4}} u \in L^2(\mathbb{R}^3),\ V_1^{\frac{1}{2}} u \in L^2(\mathbb{R}^3) \} , \end{aligned}$$

is semi-bounded from below and closed. So, it defines a unique self-adjoint operator denoted by A*. In particular assumption* (**A1**) *is verified and* (**A2**) *is satisfied thanks to a Hardy type inequality (see for instance [9, Proposition D.3]). Hence the critical value* κ_{cr} *is finite and we have the inequality for any* $z, y \in H^{1/2}(\mathbb{R}^3)$,

$$|| \frac{1}{|x|} * |z|^2 z - \frac{1}{|x|} * |y|^2 y ||_{L^2(\mathbb{R}^3)} \leq C(||z||^2_{H^{1/2}(\mathbb{R}^3)} + ||y||^2_{H^{1/2}(\mathbb{R}^3)}) \, ||z - y||_{L^2(\mathbb{R}^3)} ,$$

by the weak Young inequality, Hardy inequality and the Sobolev embedding $H^{1/2}(\mathbb{R}^3) \subset L^3(\mathbb{R}^3)$*. Furthermore, Rellich's criterion shows that* A *has compact resolvent. To prove the two limits in* (**D1**)*, we use the following argument. For any* $\xi, \Phi \in C_0^\infty(\mathbb{R}^3)$ *and* $\Psi \in C_0^\infty(\mathbb{R}^3)$,

$$|\langle \Phi, \langle \xi | \otimes 1 \mathcal{S}_2 \frac{1}{|x - y|} \Psi \rangle| \leq ||\Phi||_{L^3(\mathbb{R}^3)} \, ||T\Psi||_{L^{3/2}(\mathbb{R}^3)} \leq ||\Phi||_{H^{1/2}(\mathbb{R}^3)} \, ||T\Psi||_{L^{3/2}(\mathbb{R}^3)} , \tag{30}$$

where T *is the operator given by*

$$T\Psi(y) := \int_{\mathbb{R}^3} \overline{\xi}(x) \, \frac{1}{|x - y|} \, \Psi(x, y) \, dx .$$

Using Hölder's inequality twice with the pairs (p, q), $2 < q < 3$, $\frac{3}{2} < p < 2$ *and* $(4, \frac{4}{3})$,

$$\begin{aligned} ||T\Psi(y)||^{3/2}_{L^{3/2}(\mathbb{R}^3)} \quad & \leq \quad \int_{\mathbb{R}^3} \left| |\xi|^p * \frac{1}{|\cdot|^p} \right|^{\frac{3}{2p}} \times \left(\int_{\mathbb{R}^3} |\Psi(x, y)|^q \, dx \right)^{\frac{3}{2q}} dy \\ & \leq \quad \left\| |\xi|^p * \frac{1}{|\cdot|^p} \right\|^{3/2p}_{L^{6/p}(\mathbb{R}^3)} \left(\int_{\mathbb{R}^3} ||\Psi(\cdot, y)||^2_{L^q(\mathbb{R}^3)} \, dy \right)^{\frac{3}{4}} . \end{aligned}$$

By the fractional Gagliardo-Nirenberg's inequality in [25, Corollary 2.4], we see for $0 < \alpha < 1$ *and* $q = \frac{6}{3 - \alpha}$,

$$||T\Psi(y)||^{3/2}_{L^{3/2}(\mathbb{R}^3)} \quad \leq \quad \left\| |\xi|^p * \frac{1}{|\cdot|^p} \right\|^{3/2p}_{L^{6/p}(\mathbb{R}^3)} ||\Psi(., y)||^{2(1-\alpha)}_{L^2(\mathbb{R}^3)} \; ||(-\Delta)^{\frac{1}{4}} \Psi(., y)||^{2\alpha}_{L^2(\mathbb{R}^3)} .$$

Therefore, using the inequality $a^\alpha b^{(1-\alpha)} \leq \varepsilon a + \varepsilon^{-\frac{\alpha}{1-\alpha}} b$ *for any* $\varepsilon, a, b > 0$*, we get*

$$||T\Psi(y)||^{3/2}_{L^{3/2}(\mathbb{R}^3)} \quad \leq \quad \left\| |\xi|^p * \frac{1}{|\cdot|^p} \right\|^{3/2p}_{L^{6/p}(\mathbb{R}^3)} \left(\varepsilon \, \langle \Psi, \sqrt{-\Delta_x} \Psi \rangle_{L^2(\mathbb{R}^6)} + \varepsilon^{-\frac{\alpha}{1-\alpha}} ||\Psi||^2_{L^2(\mathbb{R}^6)} \right) . \tag{31}$$

Remark that Hardy-Littlewood-Sobolev's inequality yields

$$\left\| |\xi|^p * \frac{1}{|\cdot|^p} \right\|_{L^{6/p}(\mathbb{R}^3)} \leq C \, ||\xi||^p_{L^{\frac{6p}{6-p}}} < \infty. \tag{32}$$

So the inequalities (30),(31),(32), *provide*

$$|\langle\Phi, \langle\xi| \otimes (A+1)^{-\frac{1}{2}}\mathcal{S}_2 \frac{1}{|x-y|}(A+\lambda)^{-\frac{1}{2}} \otimes 1\Psi\rangle| \leq C\big[\varepsilon + \frac{\varepsilon^{-\frac{\alpha}{1-\alpha}}}{\lambda}\big] \,||\xi||^p_{L^{\frac{6p}{6-p}}} \,||\Phi||_{L^2(\mathbb{R}^3)} \,||\Psi||_{L^2(\mathbb{R}^6)} \,.$$

This proves the first limit when $\lambda \to \infty$, the second one is similar and it is left to the reader.
The global well-posedness in $Q(A)$, conservation of energy and charge of the semi-relativistic Hartree equation

$$\begin{cases} i\partial_t z = \sqrt{-\Delta + m^2}\, z + V(x)z + \dfrac{\lambda}{|x|} * |z|^2 z \\ z_{|t=0} = z_0. \end{cases}$$

are proved in [31, Theorem 4] for all $\kappa \geq 0$. The arguments used here extend also to non-relativistic systems of bosons with a critical interaction $W(x-y) = \frac{\kappa}{|x-y|^2}$, with $\kappa \in [0, \kappa_{cr})$, since Assumptions (**A1**)-(**A2**)-(**D1**)-(**C1**) *hold true . Under the condition $0 > \kappa > -\kappa_{cr}$, [31, Theorem 4] ensures the existence and uniqueness of strong solutions for small initial data. Therefore, assumption* **(C2)** *still holds true and Theorem 2.3 can also be applied in this case. Actually, note that the upper bound κ_{cr} can be removed. Indeed this limitation comes from assumption* (**A2**) *and the perturbation strategy using KLMN Theorem for the many-body Hamiltonian. However, using the Friedrichs extension Theorem leads to several changes and some difficulties in the proof of the mean field approximation.*

3 Properties of the Quantum Dynamics

In this section we show that under the assumptions (**A1**)-(**A2**) the quadratic form (11) defines a unique self-adjoint operator H_N. Thereafter, a useful regularity property of the related quantum dynamics is stated in Proposition 3.5.

3.1 Selfadjointness

Remember that the quadratic form q satisfies (**A2**) and $q_{i,j}^{(n)}$, q_N are defined respectively by (10) and (11).

Lemma 3.1. *Assume* (**A1**)-(**A2**). *Then, for any $1 \leq i < j \leq n$, $q_{i,j}^{(n)}$ extends to a symmetric quadratic form on $Q(A_i + A_j) \subset \otimes^n \mathcal{Z}_0$. Moreover, for any $\Phi \in Q(A_i + A_j)$,*

$$|q_{i,j}^{(n)}(\Phi^{(n)}, \Phi^{(n)})| \quad \leq \quad a\,\langle\Phi^{(n)}, A_i + A_j\, \Phi^{(n)}\rangle + b\,\|\Phi^{(n)}\|^2_{\otimes^n\mathcal{Z}_0} \,. \tag{33}$$

Proof. Once the estimate (33) is proved for any $\Phi^{(n)} \in \otimes^{alg,n}Q(A)$, the extension of $q_{i,j}^{(n)}$ to the domain $Q(A_i + A_j)$ is straightforward since $\otimes^{alg,n}Q(A)$ is a form core for $A_i + A_j$. A simple computation yields for any $\Phi^{(n)}, \Psi^{(n)} \in \otimes^{alg,n}Q(A)$,

$$q_{i,j}^{(n)}(\Phi^{(n)}, \Psi^{(n)}) = q_{1,2}^{(n)}(\Pi_{(i,j)}\Phi^{(n)}, \Pi_{(i,j)}\Psi^{(n)}) \,, \tag{34}$$

where $\Pi_{(i,j)}$ is the interchange operator defined in (6) with $\sigma = (i,j)$ is the particular permutation

$$(i,j) = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & j & \cdots & n \\ i & j & \cdots & 1 & \cdots & 2 & \cdots & n \end{pmatrix}.$$

Moreover, one remarks that

$$\langle\Pi_{(i,j)}\Phi^{(n)}, A_1 + A_2\, \Pi_{(i,j)}\Psi^{(n)}\rangle = \langle\Phi^{(n)}, A_i + A_j\, \Psi^{(n)}\rangle \,.$$

Hence, it is enough to prove (33) for $i = 1$ and $j = 2$ and $\Phi^{(n)} \in \otimes^{alg,n}Q(A)$. Let $\{e_k\}_{k\in\mathbb{N}}$ be an O.N.B of $\mathcal{Z}_0$ such that $e_k \in Q(A)$ for all $k \in \mathbb{N}$. For $r \in \mathbb{N}^n$, $r = (r_1, \cdots, r_n)$, we denote

$$e(r) := e_{r_1} \otimes \cdots \otimes e_{r_n} \in \otimes^n \mathcal{Z}_0 \,.$$

Remark that $\{e(r)\}_{r\in\mathbb{N}^n}$ is an O.N.B of $\otimes^n \mathcal{Z}_0$ and for any $\Phi^{(n)} \in \otimes^{alg,n} Q(A)$ one can write $\Phi^{(n)} = \sum_{r\in\mathbb{N}^n} \lambda(r)e(r)$ (we may assume without loss of generality that the sum is finite). Hence

$$\begin{aligned}
|q_{1,2}^{(n)}(\Phi^{(n)},\Phi^{(n)})| &= |\sum_{r,s\in\mathbb{N}^n} \overline{\lambda(r)}\lambda(s)\; q_{1,2}^{(n)}\big(e(r),e(s)\big)| \\
&\leq \left|\sum_{r_3,\cdots,r_n} q\left(\sum_{r_1,r_2} \lambda(r_1,r_2,r_3,\cdots,r_n)e_{r_1}\otimes e_{r_2}\,;\sum_{s_1,s_2}\lambda(s_1,s_2,r_3,\cdots,r_n)e_{s_1}\otimes e_{s_2}\right)\right| \\
&\leq a \sum_{r_3,\cdots,r_n} \langle \sum_{r_1,r_2} \lambda(r_1,r_2,r_3,\cdots,r_n)e_{r_1}\otimes e_{r_2}; A_1+A_2 \sum_{s_1,s_2}\lambda(s_1,s_2,r_3,\cdots,r_n)e_{s_1}\otimes e_{s_2}\rangle \\
&+ b\sum_{r_1,r_2} |\lambda(r_1,r_2,r_3,\cdots,r_n)|^2 \\
&\leq a\,\langle \Phi^{(n)}, A_1+A_2\,\Phi^{(n)}\rangle + b\,\|\Phi^{(n)}\|^2_{\otimes^n\mathcal{Z}_0}\,.
\end{aligned}$$

The second inequality follows using (**A2**). □

Remarks 3.2. *A consequence of the last proof is that for any $\Psi^{(N)},\Phi^{(N)} \in \vee^{alg,N}Q(A) = \mathcal{S}_N \otimes^{alg,N} Q(A)$,*

$$q_{i,j}^{(N)}(\Psi^{(N)},\Phi^{(N)}) = q_{1,2}^{(N)}(\Psi^{(N)},\Phi^{(N)})\,.$$

Lemma 3.3. *Assume* (**A1**)-(**A2**)*. Then q_N extends to a symmetric quadratic form on $Q(H_N^0) \subset \vee^N\mathcal{Z}$. Moreover, for any $\Psi^{(N)} \in Q(H_N^0)$,*

$$|q_N(\Psi^{(N)},\Psi^{(N)})| \quad\leq\quad a\,\langle\Psi^{(N)},H_N^0\Psi^{(N)}\rangle + bN\|\Psi^{(N)}\|^2_{\vee^N\mathcal{Z}_0}\,. \tag{35}$$

Proof. As in the previous lemma, it is enough to prove the inequality (35) for any $\Psi \in \vee^{alg,N}Q(A)$. Lemma 3.1 with Remark 3.2 yield the estimate:

$$\begin{aligned}
|q_N(\Psi^{(N)},\Psi^{(N)})| \quad &= \quad \frac{N(N-1)}{2N}|q_{1,2}^{(N)}(\Psi^{(N)},\Psi^{(N)})| \\
&\leq \quad \frac{N}{2}\left[a\langle\Psi^{(N)},A_1+A_2\Psi^{(N)}\rangle + b\|\Psi^{(N)}\|^2_{\vee^N\mathcal{Z}_0}\right].
\end{aligned}$$

Using the fact that $\langle\Psi^{(N)},A_1+A_2\Psi^{(N)}\rangle = \frac{2}{N}\langle\Psi^{(N)},H_N^0\Psi^{(N)}\rangle$, we obtain the claimed inequality. □

The lemma above allows to use the KLMN Theorem [40, Theorem X.17] since q_N is a small perturbation in the sense of quadratic forms of H_N^0 and therefore one obtains the selfadjointness of H_N.

Proposition 3.4 (Self-adjoint realization of H_N)**.** *Assume* (**A1**)-(**A2**)*, then there exists a unique self-adjoint operator H_N with $Q(H_N) = Q(H_N^0)$ satisfying for any $\Psi^{(N)},\Phi^{(N)} \in Q(H_N^0)$*

$$\langle\Psi^{(N)},H_N\Phi^{(N)}\rangle = \langle\Psi^{(N)},H_N^0\Phi^{(N)}\rangle + q_N(\Psi^{(N)},\Phi^{(N)}).$$

3.2 Invariance property

A straightforward consequence of Proposition 3.4 is that the form domain $Q(H_N^0)$ is invariant with respect to the dynamics of H_N. However, we would like to have a quantitative uniform bound on $\langle\Psi_t^{(N)},H_N^0\Psi_t^{(N)}\rangle$ for every $t\in\mathbb{R}$. Here

$$\Psi_t^{(N)} := e^{-itH_N}\Psi^{(N)}\,.$$

Proposition 3.5 (Propagation of states on $Q(H_N^0)$)**.** *Let $\Psi^{(N)} \in Q(H_N^0)$ such that $\|\Psi^{(N)}\|_{\vee^N\mathcal{Z}_0} = 1$ and satisfying:*

$$\exists C>0,\,\forall N\in\mathbb{N},\,\langle\Psi^{(N)},H_N^0\Psi^{(N)}\rangle \leq CN.$$

Then there exists a constant $C_{a,b}>0$ independent of N such that for any $t\in\mathbb{R}$ and $N\in\mathbb{N}$,

$$\langle\Psi_t^{(N)},H_N^0\Psi_t^{(N)}\rangle \leq C_{a,b}N.$$

Proof. Since $0 < a < 1$ the inequality $\pm q_N \leq aH_N^0 + bN$ implies that $H_N^0 \leq \frac{1}{1-a} H_N + \frac{b}{1-a} N$ in the form sense. Let $\Psi^{(N)} \in Q(H_N^0)$ then for any $t \in \mathbb{R}$,

$$\begin{aligned} \langle \Psi_t^{(N)}, H_N^0 \Psi_t^{(N)} \rangle &\leq \frac{1}{1-a} \langle \Psi_t^{(N)}, H_N \Psi_t^{(N)} \rangle + \frac{b}{1-a} N \\ &\leq \frac{1+a}{1-a} \langle \Psi^{(N)}, H_N^0 \Psi^{(N)} \rangle + \frac{2b}{1-a} N \\ &\leq \frac{(1+a)C + 2b}{1-a} N \,. \end{aligned}$$

The second inequality follows using the fact that $\langle \Psi_t^{(N)}, H_N \Psi_t^{(N)} \rangle = \langle \Psi^{(N)}, H_N \Psi^{(N)} \rangle$ and Lemma 3.3. □

4 Duhamel's formula

The main result provided by Theorem 2.3 is the identification of the Wigner measures of time-evolved states $\varrho_N(t)$. According to the Definition 2.2 of Wigner measures one needs simply to compute the limit when $N \to \infty$ of

$$\mathcal{I}_N(t) := \mathrm{Tr}[\varrho_N(t)\, \mathcal{W}(\sqrt{2}\pi\xi)] = \langle \Psi_t^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi) \Psi_t^{(N)} \rangle \,.$$

This task may seems quite simple but since the quantum dynamics are non trivial it is unlikely that one can compute explicitly the above limits. Therefore, it seems reasonable to rely on the dynamical properties of $\mathcal{I}_N(t)$ as for non-homogenous PDE and write a Duhamel's formula satisfied by $\mathcal{I}_N(t)$. The point here is that all the possible limits of $\mathcal{I}_N(t)$ have to satisfy a limiting integral equation. And if one can solve the latter equation then it is possible to identify the Wigner measures of $\varrho_N(t)$. This strategy was introduced in [9] for Schrödinger dynamics with singular potential. Here we improve it and extend it to a more general setting.

4.1 Commutator computation

In order to derive the aforementioned Duhamel's formula, we differentiate the quantity $\mathcal{I}_N(t)$ with respect to time. This roughly leads to the analysis of the commutator $[\mathcal{W}(\sqrt{2}\pi\xi), H_N - H_N^0]$. Since the Weyl operator do not conserve the number of particles the latter quantity has to be expanded in the symmetric Fock space. To handle this computation efficiently, we use the Wick quantization procedure explained in Appendix A and rely particularly in the properties of the class of symbols $\mathcal{Q}_{p,q}(A)$. We suggest the reading of Appendix A before going through this subsection.
Recall that $\mathfrak{Q}_n = Q(H_n^0)$ is a Hilbert space equipped with the inner product (64). The class of monomials $\mathcal{Q}_{p,q}(A)$ is defined by (65) and the energy functional satisfies:

$$h(z) = \langle z, Az \rangle + \frac{1}{2} q(z^{\otimes 2}, z^{\otimes 2}) \in Q_{1,1}(A) + Q_{2,2}(A) \,,$$

with the following relation holding for all $\Psi^{(N)}, \Phi^{(N)} \in Q(H_N^0)$,

$$\langle \Psi^{(N)}, H_N \Phi^{(N)} \rangle = \langle \Psi^{(N)}, \varepsilon^{-1} h^{Wick} \Phi^{(N)} \rangle \,, \qquad \text{when} \qquad \varepsilon = \frac{1}{N} \,.$$

The above identity stresses the relationship between the many-body Hamiltonian H_N and the Wick quantization of the energy functional $h(z)$. It allows to exploit the general properties of Wick calculus while we deal with the dynamics of H_N.
We define the following monomial q_s for any $z \in Q(A)$, $s \in \mathbb{R}$,

$$q_s(z) := \frac{1}{2} q\big((e^{-isA}z)^{\otimes 2}, (e^{-isA}z)^{\otimes 2}\big) = \frac{1}{2} \langle (e^{-isA}z)^{\otimes 2}, \tilde{q}\, (e^{-isA}z)^{\otimes 2} \rangle \,, \tag{36}$$

and check that under the assumption (**A2**),

$$q_s \in \mathcal{Q}_{2,2}(A) \qquad \text{with} \qquad \tilde{q}_s = \frac{1}{2}\, e^{isA} \otimes e^{isA} \mathcal{S}_2\, \tilde{q} \mathcal{S}_2\, e^{-isA} \otimes e^{-isA} \in \mathcal{L}(\mathfrak{Q}_2, \mathfrak{Q}_2') \,.$$

A simple computation yields for any $z \in Q(A)$ and $\xi \in Q(A)$,

$$q_s(z + i\varepsilon\pi\xi) - q_s(z) = \sum_{j=1}^{4} \varepsilon^{j-1} q_j(\xi, s) \,,$$

with the monomials $(q_j(\xi,s)[z])_{j=1,2,3,4}$ defined by:

$$q_1(\xi,s)[z] = -\pi\,\mathrm{Im}\, q(z_s^{\otimes 2}, \mathcal{S}_2\,\xi_s\otimes z_s)\,, \qquad q_2(\xi,s)[z] = -\frac{\pi^2}{2}\mathrm{Re}\, q(z_s^{\otimes 2},\xi_s^{\otimes 2}) + 2\pi^2 q(\mathcal{S}_2\xi_s\otimes z_s, \mathcal{S}_2\xi_s\otimes z_s)\,,$$
$$q_3(\xi,s)[z] = \pi^3\mathrm{Im}\, q(\xi_s^{\otimes 2}, \mathcal{S}_2\,\xi_s\otimes z_s)\,, \qquad q_4(\xi,s)[z] = \frac{\pi^4}{4}\, q(\xi_s^{\otimes 2},\xi_s^{\otimes 2}), \tag{37}$$

and the notation:

$$\xi_s := e^{-isA}\xi\,, \qquad z_s := e^{-isA}z\,.$$

Lemma 4.1. *Assume* (**A1**)-(**A2**)*, then one checks that*

$$q_1(\xi,s)[z] \in \mathcal{Q}_{2,1}(A) + \mathcal{Q}_{1,2}(A)\,, \qquad q_2(\xi,s)[z] \in \mathcal{Q}_{2,0}(A) + \mathcal{Q}_{0,2}(A) + \mathcal{Q}_{1,1}(A)\,,$$
$$q_3(\xi,s)[z] \in \mathcal{Q}_{1,0}(A) + \mathcal{Q}_{0,1}(A)\,, \qquad q_4(\xi,s)[z] \in \mathcal{Q}_{0,0}(A).$$

Proof. This result is a straightforward consequence of Proposition A.3 (iv). However for reader convenience we provide a direct proof. Remark that $q_1(\xi,s)[z]$ is a linear combination of two conjugate monomials. So it is enough to check that $q(z^{\otimes 2},\xi\otimes z)\in\mathcal{Q}_{1,2}(A)$. In fact, we have

$$\begin{aligned} b(z) = q(z^{\otimes 2},\xi\otimes z) &= \langle z^{\otimes 2}, \mathcal{S}_2\,\tilde q\,\xi\otimes z\rangle \\ &= \langle z^{\otimes 2}, \mathcal{S}_2\,\tilde q\,(|\xi\rangle\otimes 1)\, z\rangle\,. \end{aligned}$$

This implies that there exists a unique operator $\tilde b = \mathcal{S}_2\,\tilde q\,|\xi\rangle\otimes 1$ such that for any $z\in Q(A)$,

$$b(z) = \langle z^{\otimes 2}, \tilde b\, z\rangle\,.$$

Moreover $\tilde b\in\mathcal{L}(\mathfrak{Q}_1,\mathfrak{Q}_2')$ (here $\mathfrak{Q}_n = Q(H_n^0)$) since $\xi\in Q(A)$ and

$$\left((A_1+A_2+1)^{-\frac12}\tilde q\,(A+1)^{-\frac12}\otimes(A+1)^{-\frac12}\right)|(A+1)^{\frac12}\xi\rangle\otimes 1 \quad \in\mathcal{L}(\mathcal{Z}_0,\otimes^2\mathcal{Z}_0)\,.$$

Hence $b\in\mathcal{Q}_{1,2}(A)$ and $\bar b\in\mathcal{Q}_{2,1}(A)$ according to Proposition (A.3) (i). ◻

Proposition 4.2. *For $\xi\in Q(A)$ and $\varepsilon = \frac{1}{N}$, we have the following equality in the sense of quadratic forms on $Q(H_N^0)$,*

$$\frac{1}{\varepsilon}\left[q_s^{Wick}, \mathcal{W}(\sqrt{2}\pi\xi)\right] = \mathcal{W}(\sqrt{2}\pi\xi)\big[\sum_{j=1}^{4}\varepsilon^{j-1}q_j(\xi,s)^{Wick}\big], \tag{38}$$

where $q_j(\xi,s)$, $j=1,2,3,4$, are the monomials defined in (37) *and q_s is given by* (36)*.*

Proof. This follows by applying Proposition A.3 (v). ◻

4.2 Integral equation

Let $(\Psi^{(N)})_{N\in\mathbb{N}}$ be a sequence of normalized vectors in $Q(H_N^0)\subset\bigvee^N\mathcal{Z}_0$ satisfying the hypothesis of Theorem 2.3. The time evolved state is

$$\varrho_N(t) := |\Psi_t^{(N)}\rangle\langle\Psi_t^{(N)}| \quad \text{where} \quad \Psi_t^{(N)} := e^{-itH_N}\Psi^{(N)}\,.$$

Actually, it is convenient to work within the interaction representation

$$\widetilde\varrho_N(t) := |\widetilde\Psi_t^{(N)}\rangle\langle\widetilde\Psi_t^{(N)}| \quad \text{where} \quad \widetilde\Psi_t^{(N)} := e^{itH_N^0}e^{-itH_N}\Psi^{(N)}\,. \tag{39}$$

Our aim in this subsection is to write an integral equation (or Duhamel's formula) satisfied by the map

$$t\mapsto\mathcal{J}_N(t) := \mathrm{Tr}[\widetilde\varrho_N(t)\,\mathcal{W}(\sqrt{2}\pi\xi)] = \langle\widetilde\Psi_t^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)\,\widetilde\Psi_t^{(N)}\rangle\,, \tag{40}$$

and to put it in a convenient form in order to carry on the limit $N\to\infty$.

Proposition 4.3. *Assume* (**A1**)-(**A2**) *and consider a sequence $(\Psi^{(N)})_{N\in\mathbb{N}}$ of normalized vectors in $Q(H_N^0)$. Then for any $\xi\in D(A)$ the map $t\in\mathbb{R}\mapsto\mathcal{J}_N(t)$ defined in* (40) *is C^1 and satisfies for $\varepsilon=\frac{1}{N}$ and all $t\in\mathbb{R}$,*

$$\mathcal{J}_N(t) = \mathcal{J}_N(0) + i\int_0^t \langle\widetilde\Psi_s^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)\Big[\sum_{j=1}^{4}\varepsilon^{j-1}\Big(q_j(\xi,s)\Big)^{Wick}\Big]\widetilde\Psi_s^{(N)}\rangle\, ds, \tag{41}$$

where $q_j(\xi,s)$, $j=1,\cdots,4$, are the monomials given in (37)*.*

Proof. By Stone's theorem one can see that $\mathcal{J}_N(t)$ is continuously differentiable since $\Psi^{(N)} \in Q(H_N) = Q(H_N^0)$. So one obtains

$$i\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_N(t) = \langle \widetilde{\Psi}_t^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)e^{itH_N^0}(H_N - H_N^0)e^{-itH_N}\Psi^{(N)}\rangle - \langle e^{itH_N^0}(H_N - H_N^0)e^{-itH_N}\Psi^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)\widetilde{\Psi}_t^{(N)}\rangle .$$

Using the fact that $\varepsilon^{-1}q^{Wick}_{|\vee^N\mathcal{Z}_0} = H_N - H_N^0 = q_N$ in the sense of quadratic forms on $Q(H_N^0)$ and Proposition A.3, we see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_N(t) &= \langle -\frac{i}{\varepsilon}e^{itH_N^0}q^{Wick}e^{-itH_N}\Psi^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)\tilde{\Psi}_t^{(N)}\rangle + \langle \tilde{\Psi}_t^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi) - \frac{i}{\varepsilon}e^{itH_N^0}q^{Wick}e^{-itH_N}\Psi^{(N)}\rangle \\ &= \frac{i}{\varepsilon}\langle \widetilde{\Psi}_t^{(N)}, \left[q_t^{Wick}, \mathcal{W}(\sqrt{2}\pi\xi)\right]\widetilde{\Psi}_t^{(N)}\rangle ,\end{aligned}$$

where $q_t(z) = \frac{1}{2}q(z_t^{\otimes^2}, z_t^{\otimes^2}) \in \mathcal{Q}_{2,2}(A)$. The commutator and the duality bracket in the last equations make sense since $\mathcal{W}(\sqrt{2}\pi\xi)\widetilde{\Psi}_t^{(N)} \in Q(d\Gamma(A) + \mathbf{N})$ by Proposition A.4. So, the N^{th} component $\left[\mathcal{W}(\sqrt{2}\pi\xi)\widetilde{\Psi}_t^{(N)}\right]^{(N)}$ belongs to $Q(H_N^0)$. Now, we conclude by applying Proposition 4.2. ☐

5 Convergence arguments

We have established in the previous section an integral equation (41) satisfied by the quantity $\mathcal{J}_N(t)$. Here we consider its limit when $N \to \infty$. The main steps are the analysis of $\partial_t\mathcal{J}_N(t)$ and the extraction of subsequences $(N_k)_{k\in\mathbb{N}}$ that would lead to a convergent integral equation for all times. This is achieved under the assumptions (**D1**) and (**D2**).

5.1 Convergence of $\partial_t\mathcal{J}_N(t)$

The following property is crucial for the proof of convergence.

Proposition 5.1. *Let $\{\varrho_N = |\Psi^{(N)}\rangle\langle\Psi^{(N)}|\}_{N\in\mathbb{N}^*}$ be a sequence of normal states on $\vee^N\mathcal{Z}_0$ such that $\mathcal{M}(\varrho_N, N\in\mathbb{N}) = \{\mu\}$ and*

$$\exists C > 0, \forall N \in \mathbb{N},\ \langle \Psi^{(N)}, H_N^0\Psi^{(N)}\rangle \leq CN . \tag{42}$$

Assume (**A1**)-(**A2**) *and suppose that either* (**D1**) *or* (**D2**) *is true, then for any $\xi \in Q(A)$ and for every $s \in \mathbb{R}$,*

$$\lim_{\substack{N\to+\infty\\ N\varepsilon=1}}\langle \Psi^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)\,[q_1(\xi,s)]^{Wick}\,\Psi^{(N)}\rangle = \int_{\mathcal{Z}_0} e^{2i\pi\mathrm{Re}\langle\xi,z\rangle}q_1(\xi,s)[z]\,d\mu(z)\,, \tag{43}$$

where $z_s = e^{-isA}z$, $\xi_s = e^{-isA}z$ and $q_1(\xi,s)[z] = -\pi\,\mathrm{Im}\,q(z_s^{\otimes 2}, \mathcal{S}_2\,\xi_s\otimes z_s)$.

Proof. For simplicity we assume $s = 0$ since the proof goes exactly the same when $s \neq 0$. The following expression holds for any $\xi, z \in Q(A)$,

$$2q_1(\xi,0)[z] = -2\pi\,\mathrm{Im}\,q(z^{\otimes 2}, \mathcal{S}_2\,\xi\otimes z) = i\pi B_1(z) - i\pi B_2(z),$$

with

$$B_1(z) = \langle \xi\otimes z, \mathcal{S}_2\,\tilde{q}z^{\otimes 2}\rangle, \quad B_2(z) = \langle z^{\otimes 2}, \tilde{q}\,\mathcal{S}_2\,(\xi\otimes z)\rangle.$$

By the assumption (**A2**), the two symbols B_1 and B_2 belong to $\mathcal{Q}_{2,1}(A)$ and $\mathcal{Q}_{1,2}(A)$ respectively with

$$\tilde{B}_1 = \langle\xi|\otimes 1\ \mathcal{S}_2\ \tilde{q}\ \mathcal{S}_2 \in \mathcal{L}(\mathfrak{Q}_2, \mathfrak{Q}_1')\,, \qquad \tilde{B}_2 = \mathcal{S}_2\ \tilde{q}\ \mathcal{S}_2\ |\xi\rangle\otimes 1\ \in \mathcal{L}(\mathfrak{Q}_1, \mathfrak{Q}_2')\,,$$

and for any $z \in Q(A)$, $B_1(z) = \langle z, \tilde{B}_1 z^{\otimes 2}\rangle$ and $B_2(z) = \langle z^{\otimes 2}, \tilde{B}_2 z\rangle$ with the property $\overline{B_1(z)} = B_2(z)$. We will use an approximation argument. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\chi(x) = 1$ if $|x| \leq 1$, $\chi(x) = 0$ if $|x| \geq 2$ and $0 \leq \chi \leq 1$. We denote for $m \in \mathbb{N}^*$, $\chi_m(x) = \chi(\frac{x}{m})$ and $H_1^0 = A$, $H_2^0 = A_1 + A_2$ and set

$$\tilde{B}_{1,m} := \chi_m(H_1^0)\,\tilde{B}_1\,\chi_m(H_2^0) \in \mathcal{L}(\vee^2\mathcal{Z}_0, \mathcal{Z}_0)\,, \qquad \tilde{B}_{2,m} := \chi_m(H_2^0)\,\tilde{B}_2\,\chi_m(H_1^0) \in \mathcal{L}(\mathcal{Z}_0, \vee^2\mathcal{Z}_0)\,,$$

and

$$B_{1,m}(z) = \langle z, \tilde{B}_{1,m}z^{\otimes 2}\rangle\,, \qquad B_{2,m}(z) = \langle z^{\otimes 2}, \tilde{B}_{2,m}z\rangle\,.$$

Since (**D1**) says that A has compact resolvent and both operators $(H_1^0+1)^{-\frac{1}{2}}\tilde{B}_1(H_2^0+1)^{-\frac{1}{2}}$ and $(H_2^0+1)^{-\frac{1}{2}}\tilde{B}_2(H_1^0+1)^{-\frac{1}{2}}$ are either compact or bounded, we see that $B_{j,m}$ are compact operators once we assume (**D1**) or (**D2**). We now write the following inequalities for $j=1,2$,

$$|\langle\Psi^{(N)},\mathcal{W}(\sqrt{2}\pi\xi)\,B_j^{Wick}\,\Psi^{(N)}\rangle-\mu(e^{2i\pi\mathrm{Re}\langle\xi,z\rangle}B_j(z))|\leq\mathcal{A}_j^{(m)}+\mathcal{B}_j^{(m)}+\mathcal{C}_j^{(m)}, \tag{44}$$

where

$$\mathcal{A}_j^{(m)}=|\langle\Psi^{(N)},\mathcal{W}(\sqrt{2}\pi\xi)[B_j-B_{j,m}]^{Wick}\Psi^{(N)}\rangle|,$$

$$\mathcal{B}_j^{(m)}=|\langle\Psi^{(N)},\mathcal{W}(\sqrt{2}\pi\xi)\,B_{j,m}^{Wick}\Psi^{(N)}\rangle-\mu(e^{2i\pi\mathrm{Re}\langle\xi,z\rangle}B_{j,m}(z))|,$$

and

$$\mathcal{C}_j^{(m)}=|\mu(e^{2i\pi\mathrm{Re}\langle\xi,z\rangle}B_{j,m}(z))-\mu(e^{2i\pi\mathrm{Re}\langle\xi,z\rangle}B_j(z))|.$$

To prove the limit (43), we show that all the terms $\mathcal{A}_j^{(m)},\mathcal{B}_j^{(m)},\mathcal{C}_j^{(m)}$ can be made arbitrary small for all N larger enough by choosing a convenient $m\in\mathbb{N}$.

The term $\mathcal{C}_j^{(m)}$: By dominated convergence theorem the quantity $\mathcal{C}_j^{(m)}$ tends to 0 when $m\to\infty$ for $j=1,2$. In fact $B_{j,m}(z)$ converges to $B_j(z)$ for all $z\in Q(A)$ since $s-\lim\chi_m(H_j^0)=\mathrm{Id}$. Moreover, we have for some $C'>0$ and any $z\in Q(A)$,

$$|B_{j,m}(z)|\leq C'\|\xi\|_{Q(A)}\,\|z\|_{Q(A)}^2\,\|z\|_{\mathcal{Z}_0}\,, \tag{45}$$

since $B_{j,m}$ are in $\mathcal{Q}_{1,2}(A)$ or $\mathcal{Q}_{2,1}(A)$ and by Proposition A.6 we get the a priori estimate:

$$\int_{\mathcal{Z}_0}\|z\|_{Q(A)}^2\|z\|_{\mathcal{Z}_0}\,d\mu(z)\leq C\,. \tag{46}$$

The term $\mathcal{B}_j^{(m)}$: Since $\tilde{B}_{j,m}$ are compact operators for $j=1,2$ and any $m\in\mathbb{N}^*$, the quantity $\mathcal{B}_j^{(m)}\to 0$ when $N\to\infty$ owing to result proved in [6, Theorem 6.13] and [6, Corollary 6.14].

The term $\mathcal{A}_j^{(m)}$: We consider only $j=1$ since the case $j=2$ is quite similar. We write for any $z\in Q(A)$,

$$B_1(z)-B_{1,m}(z)=\langle z,(1-\chi_m(H_1^0))\tilde{B}_1\,z^{\otimes 2}\rangle+\langle z,\chi_m(H_1^0)\tilde{B}_1(1-\chi_m(H_2^0))\,z^{\otimes 2}\rangle=:\mathcal{U}_1(z)+\mathcal{U}_2(z)\,,$$

and check that $\mathcal{U}_1,\mathcal{U}_2\in\mathcal{Q}_{2,1}(A)$. Let $\Phi^{(N-1)}=[\mathcal{W}(\sqrt{2}\pi\xi)\Psi^{(N)}]^{(N-1)}$ be the $(N-1)^{\mathrm{th}}$ component of the vector $\mathcal{W}(\sqrt{2}\pi\xi)\Psi^{(N)}$ in the symmetric Fock space $\Gamma_s(\mathcal{Z}_0)$. By Proposition A.4 we see that $\Phi^{(N-1)}\in Q(H_{N-1}^0)$. So, one obtains

$$\mathcal{A}_1^{(m)}=\underbrace{\langle\Phi^{(N-1)},\mathcal{U}_1^{Wick}\,\Psi^{(N)}\rangle}_{(1)}+\underbrace{\langle\Phi^{(N-1)},\mathcal{U}_2^{Wick}\,\Psi^{(N)}\rangle}_{(2)}.$$

Now estimate each term. Let denote $\overline{\chi}_m=1-\chi_m$ then for $\lambda>0$ and $\varepsilon=\frac{1}{N}$,

$$\begin{aligned}\left|(1)\right| &= \left|\langle\Phi^{(N-1)},\varepsilon^{3/2}\sqrt{N(N-1)^2}\,\mathcal{S}_{N-1}\,\overline{\chi}_m(H_1^0)\tilde{B}_1\otimes 1^{(N-2)}\,\Psi^{(N)}\rangle\right|\\ &\leq \left|\langle\overline{\chi}_m(H_1^0)\otimes 1^{(N-2)}\Phi^{(N-1)},\,\tilde{B}_1\otimes 1^{(N-2)}\,\Psi^{(N)}\rangle\right|\\ &\leq \alpha(\lambda)\left\|(H_1^0+\lambda)^{1/2}\overline{\chi}_m(H_1^0)\otimes 1^{(N-2)}\Phi^{(N-1)}\right\|\,\left\|(H_2^0+1)^{1/2}\otimes 1^{(N-2)}\Psi^{(N)}\right\|,\end{aligned}$$

where

$$\alpha(\lambda)=\left\|(H_1^0+\lambda)^{-1/2}\tilde{B}_1(H_2^0+1)^{-1/2}\right\|_{\mathcal{L}(\bigvee^2\mathcal{Z}_0,\mathcal{Z}_0)}\to 0,\qquad\text{when }\lambda\to\infty\,.$$

Remark that the spectral theorem yields,

$$\forall m\in\mathbb{N}^*,\ \|\overline{\chi}_m(A)\,(A+1)^{-\frac{1}{2}}\|_{\mathcal{L}(\mathcal{Z}_0)}^2\leq\frac{1}{m}.$$

So using the assumption (42), the symmetry of $\Phi^{(N-1)}$ and Proposition A.4, one obtains

$$\left\|(H_1^0+\lambda)^{1/2}\overline{\chi}_m(H_1^0)\otimes 1^{(N-2)}\Phi^{(N-1)}\right\|\leq C_1\sqrt{1+\frac{\lambda}{m}}\,,$$

form some $C_1 > 0$ independent of N. Hence $|(1)| \lesssim \alpha(\lambda)\sqrt{1+\frac{\lambda}{m}}$ and if we choose $\lambda = m$ we see that $|(1)| \to 0$ when $m \to \infty$.
Similar computation yields for λ large enough

$$\Big|(2)\Big| \quad \leq \quad \beta(\lambda)\Big\|(H_1^0+\lambda)^{1/2} \otimes 1^{(N-2)}\Phi^{(N-1)}\Big\| \ \Big\|\overline{\chi}_m(H_2^0)(H_2^0+\lambda)^{1/2} \otimes 1^{(N-2)}\Psi^{(N)}\Big\|,$$

where

$$\beta(\lambda) = \Big\|(H_1^0+1)^{-1/2}\tilde{B}_1(H_2^0+\lambda)^{-1/2}\Big\|_{\mathcal{L}(\vee^2\mathcal{Z}_0,\mathcal{Z}_0)} \to 0, \qquad \text{when } \lambda \to \infty\,.$$

So by the same argument above we conclude that $|(2)| \lesssim \beta(\lambda)\sqrt{1+\frac{\lambda}{m}}$ and if we choose again $\lambda = m$ we get $|(2)| \to 0$ when $m \to \infty$.
This proves the claimed limit (43) for any $\xi \in D \subset \mathcal{Z}_0$. So we extend this result to any $\xi \in Q(A)$ by an approximation argument. In fact take for any $\xi \in Q(A)$ a sequence $(\xi_m)_{m\in\mathbb{N}}$ such that $\xi_m \to \xi$ in $Q(A)$. Write

$$\left|\langle \Psi^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)\,[q_1(\xi,0)]^{Wick}\,\Psi^{(N)}\rangle - \int_{\mathcal{Z}_0} e^{2i\pi \mathrm{Re}\langle\xi,z\rangle} q_1(\xi,0)[z]\; d\mu(z)\right| \quad \leq \quad \mathcal{A}^{(m)} + \mathcal{B}^{(m)} + \mathcal{C}^{(m)},$$

with

$$\mathcal{A}^{(m)} = \left|\langle \Psi^{(N)}, \Big(\mathcal{W}(\sqrt{2}\pi\xi) - \mathcal{W}(\sqrt{2}\pi\xi_m)\Big) q_1(\xi,0)^{Wick}\Psi^{(N)}\rangle\right|,$$

$$\mathcal{B}^{(m)} = \left|\langle \Psi^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi_m)\, q_1(\xi,0)^{Wick}\Psi^{(N)}\rangle - \mu(e^{2i\pi\mathrm{Re}\langle\xi_m,z\rangle} q_1(\xi,0)[z])\right|,$$

and

$$\mathcal{C}^{(m)} = \left|\mu(e^{2i\pi\mathrm{Re}\langle\xi_m,z\rangle} q_1(\xi,0)[z]) - \mu(e^{2i\pi\mathrm{Re}\langle\xi,z\rangle} q_1(\xi,0)[z])\right|.$$

So using Number-Weyl estimates in [6, Lemma 3.1], one shows that $\mathcal{A}^{(m)} \lesssim ||\xi - \xi_m||_{\mathcal{Z}_0}$ and hence $\mathcal{A}^{(m)} \to 0$. Now, $\mathcal{B}^{(m)} \to 0$ by the result proved above and $\mathcal{C}^{(m)} \to 0$ by (45)-(46) and the dominated convergence theorem. □

5.2 Existence of Wigner measures for all times

Wigner measures and their properties were studied in infinite dimensional spaces in [6]. A result proved in [6, Theorem 6.2] says that for any sequence of normal states $\{\tilde{\varrho}_N(t)\}_{N\in\mathbb{N}}$ as in (39) we can extract a subsequence $(N_k)_{k\in\mathbb{N}}$ such that $\tilde{\varrho}_{N_k}(t)$ has a unique Wigner measure $\tilde{\mu}_t$ according to Definition 2.2. However, the subsequence may depend in the time $t \in \mathbb{R}$. So, in order to carry on the limit on the integral equation (41) we need to extract a subsequence $(N_k)_{k\in\mathbb{N}}$ for all $t \in \mathbb{R}$ that gives $\mathcal{M}(\tilde{\varrho}_{N_k}(t), k \in \mathbb{N}) = \{\tilde{\mu}_t\}$.

Proposition 5.2. *Let $\{\varrho_N = |\Psi^{(N)}\rangle\langle\Psi^{(N)}|\}_{N\in\mathbb{N}}$ be a sequence of normal states on $\vee^N\mathcal{Z}_0$ such that*

$$\exists C > 0, \forall N \in \mathbb{N},\ \langle \Psi^{(N)}, H_N^0\Psi^{(N)}\rangle \leq CN\,,$$

and $\mathcal{M}(\varrho_N, N \in \mathbb{N}) = \{\mu_0\}$. Then for any $\xi \in Q(A)$ and for any subsequence $(N_k)_{k\in\mathbb{N}}$ there exist a family of probability measures $(\mu_t)_{t\in\mathbb{R}}$ on $\mathcal{Z}_0$ and a subsequence $(N_{k_l})_{l\in\mathbb{N}}$ such that for all $t \in \mathbb{R}$,

$$\mathcal{M}\Big(\big|e^{-itH^0_{N_{k_l}}} e^{itH_{N_{k_l}}}\,\Psi^{(N_{k_l})}\big\rangle\big\langle e^{-itH^0_{N_{k_l}}} e^{itH_{N_{k_l}}}\,\Psi^{(N_{k_l})}\big|, l \in \mathbb{N}\Big) = \{\tilde{\mu}_t\},$$

and the following Liouville equation is satisfied for any $\xi \in Q(A)$,

$$\begin{aligned}\tilde{\mu}_t(e^{2i\pi\mathrm{Re}\langle\xi,z\rangle}) &= \tilde{\mu}_0(e^{2i\pi\mathrm{Re}\langle\xi,z\rangle}) + i\int_0^t \tilde{\mu}_s(e^{2i\pi\mathrm{Re}\langle\xi,z\rangle} q_1(\xi,s)[z])\, ds \\ &= \tilde{\mu}_0(e^{2i\pi\mathrm{Re}\langle\xi,z\rangle}) + i\int_0^t \tilde{\mu}_s\big(\{q_s(z); e^{2i\pi\mathrm{Re}\langle\xi,z\rangle}\}\big) ds,\end{aligned} \tag{47}$$

with $z_s = e^{-isA}z$, $\xi_s = e^{-isA}\xi$, $q_1(\xi,s) = -\pi\,\mathrm{Im}\, q(z_s^{\otimes 2}, \mathcal{S}_2\,\xi_s \otimes z_s)$, $q_s(z) = \frac{1}{2}q(z_s^{\otimes 2}, z_s^{\otimes 2})$ and the bracket $\{b_1; b_2\}(z)$ equals to $\partial_z b_1(z)\cdot\partial_{\bar{z}}b_2(z) - \partial_z b_2(z)\cdot\partial_{\bar{z}}b_1(z)$.

Proof. The extraction of such subsequence $(N_{k_l})_{l\in\mathbb{N}}$ for all times follows by an Ascoli type argument proved in [8, Proposition 3.3]. Here we briefly check the main points. Wigner measures are identified through (14). Hence we consider the quantities:

$$G_N(t,\xi)=\langle \widetilde{\Psi}_t^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)\,\widetilde{\Psi}_t^{(N)}\rangle.$$

We wish to prove the existence of a subsequence $(N_{k_l})_{l\in\mathbb{N}}$ such that $G_{N_{k_l}}(t,\xi)$ converges for all $t\in\mathbb{R}$ and $\xi\in\mathcal{Z}_0$. For this, we exploit the regularity of the functions $G_N(t,\xi)$ with respect to t and ξ. In some sense we have to prove that the family $(G_N)_{N\in\mathbb{N}}$ is equi-continuous on bounded sets of $\mathbb{R}\times\mathcal{Z}_0$. By using Lemma 3.1 in [6] we get for $\xi,\eta\in Q(A)$,

$$\|\big[\mathcal{W}(\sqrt{2}\pi\xi)-\mathcal{W}(\sqrt{2}\pi\eta)\big](\mathbf{N}+1)^{-\frac{1}{2}}\|_{\mathcal{L}(\Gamma_s(\mathcal{Z}_0))}\lesssim\|\xi-\eta\|_{\mathcal{Z}_0}\sqrt{\|\xi\|_{\mathcal{Z}_0}^2+\|\eta\|_{\mathcal{Z}_0}^2+1}\,.$$

Therefore, the following estimate holds

$$|G_N(t,\xi)-G_N(t,\eta)|\lesssim\|\xi-\eta\|_{\mathcal{Z}_0}\sqrt{\|\xi\|_{\mathcal{Z}_0}^2+\|\eta\|_{\mathcal{Z}_0}^2+1}\,. \tag{48}$$

On the other hand by using Proposition 4.3, Proposition A.3 (iii) and Proposition A.4, we get for any $s,t\in\mathbb{R},\xi\in Q(A)$ and $\varepsilon=\frac{1}{N}$,

$$\begin{aligned}|G_N(s,\xi)-G_N(t,\xi)|\leq&\left|\int_s^t\langle\widetilde{\Psi}_r^{(N)},\mathcal{W}(\sqrt{2}\pi\xi)\sum_{j=1}^4\varepsilon^{j-1}\,q_j(\xi,r)^{Wick}\,\widetilde{\Psi}_r^{(N)}\rangle\,dr\right|\\ \lesssim&(1+\|\xi\|_{Q(A)}^4)\,|s-t|\sup_{s\leq r\leq t}\|(A_1+1)^{\frac{1}{2}}\widetilde{\Psi}_r^{(N)}\|^2_{\bigvee^N\mathcal{Z}_0}\;\lesssim\;(1+\|\xi\|_{Q(A)}^4)\,|s-t|\,.\end{aligned}$$

Hence combining (48) with the latter inequality one gets for any $\eta,\xi\in Q(A)$ and $s,t\in\mathbb{R}$,

$$|G_N(t,\xi)-G_N(s,\eta)|\lesssim|s-t|(1+\|\xi\|_{Q(A)}^4)+\|\xi-\eta\|_{\mathcal{Z}_0}\sqrt{\|\xi\|_{\mathcal{Z}_0}^2+\|\eta\|_{\mathcal{Z}_0}^2+1}\,.$$

Furthermore the uniform estimate $|G_N(t,\xi)|\leq 1$ holds true. By an Ascoli type argument as in [8, Proposition 3.3] and [9, Proposition 3.9], we see that for any sequence $(N_k)_{k\in\mathbb{N}}$, there exists a subsequence $(N_{k_l})_{l\in\mathbb{N}}$ and a family of Borel probability measures $(\tilde{\mu}_t)_{t\in\mathbb{R}}$ on $\mathcal{Z}_0$ satisfying for any $t\in\mathbb{R}$,

$$\mathcal{M}\left(|\widetilde{\Psi}_t^{(N_{k_l})}\rangle\langle\widetilde{\Psi}_t^{(N_{k_l})}|,\,l\in\mathbb{N}\right)=\{\tilde{\mu}_t\}\,.$$

Now to prove the integral equation (47), we use Proposition 4.3 with $\varepsilon=\frac{1}{N_{k_l}}$,

$$\mathcal{J}_{N_{k_l}}(t)=\mathcal{J}_{N_{k_l}}(0)+i\int_0^t\langle\widetilde{\Psi}_s^{(N_{k_l})},\mathcal{W}(\sqrt{2}\pi\xi)\big[\sum_{j=1}^4(\varepsilon^{j-1}q_j(\xi,s)^{Wick}\big]\,\widetilde{\Psi}_s^{(N_{k_l})}\rangle\,ds, \tag{49}$$

with the monomials $(q_j(\xi,s))_{j=1,2,3,4}$ given by (37). The estimates provided by Proposition A.3 (iii) and Proposition A.4 give the convergence towards 0 of the terms involving $q_j(\xi,s)^{Wick}$, $j=2,3,4$ when $l\to\infty$. Applying the Proposition 5.1 to the subsequence $|\widetilde{\Psi}_s^{(N_{k_l})}\rangle\langle\widetilde{\Psi}_s^{(N_{k_l})}|$, we obtain the claimed equation (47). Remark that in order to check the hypothesis (42) of Proposition 5.1 we have used Proposition 3.5. $\square$

6 The Liouville equation

Once Proposition 5.1 is proved we are led to the problem of solving a Liouville (continuity or transport) equation in infinite dimension which already admits measure-valued solutions. So the point is to prove uniqueness. The method we use for uniqueness here is introduced in [9] and uses some techniques from optimal transport theory initiated in the book [3].

6.1 Properties of measure-valued solutions to Liouville equation

We need some preliminaries. The sets of all Borel probability measures on $Q'(A)$ and $Q(A)$ are denoted by $\mathfrak{P}(Q'(A))$ and $\mathfrak{P}(Q(A))$ respectively. We introduce some classes of cylindrical functions on $Q'(A)$.

Denote $\mathbb{P}$ the space of finite rank orthogonal projections on $Q'(A)$. We say that a function f is in the cylindrical Schwartz space $\mathcal{S}_{cyl}(Q'(A))$ (resp. $C^{\infty}_{0,cyl}(Q'(A))$) if:

$$\exists \mathfrak{p} \in \mathbb{P},\ \exists g \in \mathcal{S}(\mathfrak{p}Q'(A))\ \big(resp.\ C^{\infty}_{0,cyl}(\mathfrak{p}Q'(A))\big),\ \forall z \in Q'(A), f(z) = g(\mathfrak{p}z).$$

The space $C^{\infty}_{0,cyl}(\mathbb{R} \times Q'(A))$ of smooth cylindrical functions with compact support on $\mathbb{R} \times Q'(A)$ will be useful too and it is defined in the same way. Denote $L_{\mathfrak{p}}(dz)$ the Lebesgue measure on the finite dimensional subspace $\mathfrak{p}Q'(A)$. The Fourier transform of functions in $\mathcal{S}_{cyl}(Q'(A))$ are given by

$$\mathcal{F}[f](\xi) = \int_{\mathfrak{p}Q'(A)} f(z) e^{-2i\pi \mathrm{Re}\langle z,\xi\rangle_{Q'(A)}} L_{\mathfrak{p}}(dz),$$

After fixing a Hilbert basis $(e_n)_{n\in\mathbb{N}^*}$, the space $Q'(A)$ as a real Hilbert space, can be equipped with a norm,

$$\|z\|_{Q'(A),w} = \sqrt{\sum_{n\in\mathbb{N}^*} \frac{|\mathrm{Re}\langle z, e_n\rangle_{Q'(A)}|^2}{n^2}}.$$

The norm $\|.\|_{Q'(A)}$ and $\|.\|_{Q'(A),w}$ topology lead to two distinct notions of narrow convergence of probability measures. On the one hand, a sequence $(\mu_n)_{n\in\mathbb{N}}$ is narrowly convergent to $\mu \in \mathfrak{P}(Q'(A))$ if

$$\lim_{n\to+\infty} \int_{Q'(A)} f(z) d\mu_n(z) = \int_{Q'(A)} f(z) d\mu(z), \tag{50}$$

for every function $f \in \mathcal{C}^0_b(Q'(A), \|.\|_{Q'(A)})$, the space of continuous and bounded real functions defined on $Q'(A)$ with the norm topology. On the other hand, a sequence $(\mu_n)_{n\in\mathbb{N}}$ is weakly narrowly convergent if the limit (50) holds for all $f \in \mathcal{C}^0_b(Q'(A), \|.\|_{Q'(A),w})$. The family of probability measures $(\tilde{\mu}_t)_{t\in\mathbb{R}}$ provided by Proposition 5.2 have uniformly bounded moments $\int_{Q'(A)} \|z\|^{2k}_{\mathcal{Z}_0} d\tilde{\mu}_t(z) \le 1$ for all $k \in \mathbb{N}$ thanks to Proposition A.6. We refer to [3, Chapter V] or [5] for a more complete presentation of those notions.

Proposition 6.1. *Let $\{|\Psi^{(N)}\rangle\langle\Psi^{(N)}|\}_{N\in\mathbb{N}}$ a sequence of normal states in $\bigvee^N \mathcal{Z}_0$ satisfying the uniform estimate:*

$$\exists C > 0\,, \forall N \in \mathbb{N},\ \langle \Psi^{(N)}, H^0_N \Psi^{(N)}\rangle \le CN\,.$$

Consider an extracted subsequence $(N_k)_{k\in\mathbb{N}}$ according to Proposition 5.2 such that for any $t \in \mathbb{R}$,

$$\mathcal{M}(|\widetilde{\Psi}^{(N_k)}_t\rangle\langle\widetilde{\Psi}^{(N_k)}_t|, k \in \mathbb{N}) = \{\tilde{\mu}_t\}\,,$$

where $\widetilde{\Psi}^{(N_k)}_t$ is given by (39). Then the Borel probability measures $\tilde{\mu}_t$ on $\mathcal{Z}_0$ satisfy:

- *(i) $\tilde{\mu}_t$ are Borel probability measures on $Q(A)$.*
- *(ii) The map $t \mapsto \tilde{\mu}_t$ is weakly narrowly continuous in $Q'(A)$.*
- *(iii) The measure $\tilde{\mu}_t$ is a weak solution to the Liouville equation*

$$\partial_t \tilde{\mu}_t + i\{q_t(z); \tilde{\mu}_t\} = 0,$$

i.e: for all $f \in C^{\infty}_{0,cyl}(\mathbb{R} \times Q'(A))$

$$\int_{\mathbb{R}} \int_{Q(A)} (\partial_t f(t,z) + i\{q_t(.), f(t,.)\}(z))\, d\tilde{\mu}_t(z)\, dt = 0\,,$$

where $z_t = e^{-itA} z$ and $q_t(z) = \frac{1}{2} q(z_t^{\otimes 2}, z_t^{\otimes 2})$.

Proof. The statement (i) is proved in [9, Proposition 3.11] when $A = -\Delta$ but the proof works without any change for a general operator A satisfying (**A1**). The proof of the statements (ii)-(iii) are also essentially the same as in [9, Proposition 3.14]. We briefly sketch here the main arguments.
(ii) *Weakly narrowly continuity:*
The characteristic function of $\tilde{\mu}_t$ as a probability measure on $Q(A)$ is given by

$$G(t,\xi) = \tilde{\mu}_t(e^{2i\pi \mathrm{Re}\langle \xi, z\rangle_{Q'(A)}}) := \tilde{\mu}_t(e^{2i\pi \mathrm{Re}\langle \xi, (A+1)^{-1} z\rangle_{\mathcal{Z}_0}}).$$

The following inequality holds as in [9, Proposition 3.11] for any $\xi, \xi' \in Q'(A)$,

$$|G(t,\xi) - G(t,\xi')| \leq \pi \|\xi - \xi'\|_{Q'(A)} \int_{\mathcal{Z}_0} \|z\|^2_{Q(A)} \, d\tilde{\mu}_t(z). \tag{51}$$

Since by Lemma 3.5 there exists a time independent constant $C' > 0$ such that $\langle \Psi_t^{(N)}, H_N^0 \Psi_t^{(N)} \rangle \leq C'N$, one obtains using Proposition A.6 the uniform estimate,

$$\int_{\mathcal{Z}_0} \|z\|^2_{Q(A)} \, d\tilde{\mu}_t(z) \leq C' \, . \tag{52}$$

Subsequently for any $\xi, \xi' \in Q'(A)$,

$$|G(t,\xi) - G(t,\xi')| \lesssim \|\xi - \xi'\|_{Q'(A)}. \tag{53}$$

On the other hand for any $\xi \in Q'(A)$ and $t, t' \in \mathbb{R}$, the following estimate holds true

$$|G(t',\xi) - G(t,\xi)| \leq \Big| \int_{t'}^{t} \tilde{\mu}_s(e^{2i\pi \mathrm{Re}\langle \xi, (A+1)^{-1} z\rangle} q_1(\xi,s)[z]) \, ds \Big| \leq (C'+1) \, |t - t'| \, \|\xi\|_{Q'(A)}, \tag{54}$$

owing to (**C1**) and Proposition A.6. Now let $g \in S_{cyl}(Q'(A))$ based on $\mathfrak{p}Q'(A)$ and

$$I_g(t) := \int_{\mathfrak{p}Q'(A)} g(z) \; d\tilde{\mu}_t(z) = \int_{\mathfrak{p}Q'(A)} \mathcal{F}[g](\xi) \; G(\xi,t) \, L_{\mathfrak{p}}(d\xi).$$

Then we easily check:

- $t \longrightarrow \mathcal{F}[g](\xi) \; G(t,\xi)$ is continuous owing to (54).
- $\xi \longrightarrow \mathcal{F}[g](\xi) \; G(t,\xi)$ is bounded by a $L_{\mathfrak{p}}(d\xi)$-integrable function.

Thus $I_g(\cdot)$ is continuous for all $g \in S_{cyl}(Q'(A))$ and the bound (52) holds true. Hence we can apply Lemma 5.12-f) in [3] and then conclude that the map $t \to \tilde{\mu}_t$ is weakly narrowly continuous in $Q'(A)$.

The Liouville equation:
Integrate the expression (47) with $\mathcal{F}[g](\xi) L_{\wp}(dz)$, hence $\forall t \in \mathbb{R}$, $\forall g \in S_{cyl}(Q'(A))$,

$$\partial_t I_g(t) = i \int_{Q(A)} \{q_t; g\}(z) d\tilde{\mu}_t(z),$$

with $q_t(z) = \frac{1}{2} q(z_t^{\otimes 2}, z_t^{\otimes 2})$. Multiplying this expression by $\phi \in C_0^\infty(\mathbb{R})$ and integrating by parts yields

$$\int_{\mathbb{R}} \int_{Q(A)} \partial_t f(t,z) + i\{q_t(.), f(t,.)\}(z) \; d\tilde{\mu}_t(z) dt = 0 \, ,$$

with $f(t,z) = g(z)\phi(t)$. To conclude, we use the density of $C_0^\infty(\mathbb{R}) \otimes^{alg} C_{0,cyl}^\infty(Q'(A))$ in $C_{0,cyl}^\infty(\mathbb{R} \times Q'(A))$. □

6.2 End of the Proof of Theorems 2.3

Proof. Assume the hypotheses of Theorem 2.3 and consider for a given time $t \in \mathbb{R}$ the family of normal states,

$$\tilde{\varrho}_N(t) = |\widetilde{\Psi}_t^{(N)}\rangle\langle\widetilde{\Psi}_t^{(N)}| = |e^{itH_N^0} e^{-itH_N} \Psi^{(N)}\rangle\langle e^{itH_N^0} e^{-itH_N} \Psi^{(N)}| \, .$$

Suppose that ν is any Wigner measure of $\tilde{\varrho}_N(t)$ then there exists a subsequence $(N_k)_{k\in\mathbb{N}}$ such that $\{\nu\} = \mathcal{M}(\varrho_{N_k}(t), k \in \mathbb{N})$ according to Definition 2.2. By Proposition 5.2 and 6.1, we can extract a subsequence $(N_{k_l})_{l\in\mathbb{N}}$ such that for all $s \in \mathbb{R}$,

$$\mathcal{M}(\tilde{\varrho}_{N_{k_l}}(s), l \in \mathbb{N}) = \{\tilde{\mu}_s\} \qquad \text{with in particular} \qquad \tilde{\mu}_t = \nu \, .$$

We know by Proposition 6.1 that $s \in \mathbb{R} \to \tilde{\mu}_s$ solves the Liouville (transport) equation (47) and by setting $\xi = (1+A)^{-1}\eta$, $\eta \in Q'(A)$, we get

$$\tilde{\mu}_t(e^{2i\pi \mathrm{Re}\langle \eta, z\rangle_{Q'(A)}}) = \tilde{\mu}_0(e^{2i\pi \mathrm{Re}\langle \eta, z\rangle_{Q'(A)}}) + i \int_0^t \tilde{\mu}_s(e^{2i\pi \mathrm{Re}\langle \eta, z\rangle_{Q'(A)}} q_1(\eta,s)[z]) \, ds, \tag{55}$$

Then we have

$$\partial_s \tilde{\mu}_s + i\{q_s(z), \tilde{\mu}_s\} = \partial_s \tilde{\mu}_s + \nabla^T (v_s(z)\tilde{\mu}_s) = 0,$$

in the weak sense (47), since for any $f \in C^\infty_{0,cyl}(\mathbb{R} \times Q'(A))$,

$$\begin{aligned} i\{q_s(.), f(s,.)\}(z) &= i\langle [\partial_{\bar z} q_0](e^{-isA}z), e^{-isA}\partial_{\bar z} f(s,z)\rangle - i\langle e^{-isA}\partial_{\bar z} f(s,z), [\partial_{\bar z} q_0](e^{-isA}z)\rangle \\ &= 2\,\mathrm{Re}\langle v_s(z), \partial_{\bar z} f(s,z)\rangle = 2\,\mathrm{Re}\langle v_s(z), \nabla_{\bar z} f(s,z)\rangle_{Q'(A)} \\ &= \mathrm{Re}\langle v_s(z), \nabla f(s,z)\rangle_{Q'(A)}, \end{aligned}$$

where $v_s(z) = -ie^{isA}[\partial_{\bar z} q_0](e^{-isA}z)$, $q_s(z) = \frac{1}{2} q(z_s^{\otimes 2}, z_s^{\otimes 2})$, $z_s = e^{-isA}z$. Here v_s has the interpretation of a velocity vector field and ∇ is the real derivative in $Q'(A)$, see [9, Lemma C.7] for more details. By Proposition 3.5, we see that for any $s \in \mathbb{R}$,

$$\langle \widetilde{\Psi}_s^{(N)}, H_N^0 \widetilde{\Psi}_s^{(N)}\rangle \leq C'N\,,$$

for some time independent constant $C' > 0$. Thus Proposition A.6 gives for any $s \in \mathbb{R}$, and for every $k \in \mathbb{N}$

$$\int_{Q(A)} \|z\|^2_{Q(A)}\, \|z\|^{2k}_{\mathcal{Z}_0}\, d\tilde{\mu}_s(z) \leq C'\,.$$

Subsequently, for any time $t \in \mathbb{R}$, $\mu_t(B_{\mathcal{Z}_0}(0,1)) = 1$. Now the abstract field equation

$$i\partial_t z = Az + [\partial_{\bar z} q_0](z),$$

can be written in the interaction representation as follows:

$$\begin{cases} \partial_t z = v_t(z) = -ie^{itA}[\partial_{\bar z} q_0](e^{-itA}z), \\ z_{|t=0} = z_0. \end{cases} \tag{56}$$

So the above equation (56) is locally well-posed in $Q(A)$, in the sense of Assumption **(C2)**. Remember that Proposition 6.1 says that the map $s \to \tilde{\mu}_s$ is weakly narrowly continuous in $Q'(A)$. Subsequently the measures $(\tilde{\mu}_s)_{s\in\mathbb{R}}$ is satisfying all the assumptions of Theorem B.1. Then we get

$$\forall s \in \mathbb{R},\ \tilde{\mu}_s = \tilde{\Phi}(s,0) * \mu_0,$$

where $\tilde{\Phi}(s,0)$ denotes the well defined flow of the equation (56). In particular one gets the equality $\nu = \tilde{\Phi}(t,0) * \mu_0$. Since ν is any Wigner measure of $(\tilde{\varrho}_N(t))_{N\in\mathbb{N}}$, one obtains

$$\mathcal{M}(\tilde{\varrho}_N(t), N \in \mathbb{N}) = \{\tilde{\Phi}(t,0) * \mu_0\}\,.$$

Back to the family of normal states,

$$\varrho_N(t) = e^{-itH_N^0}\, \tilde{\varrho}_N(t)\, e^{itH_N^0}\,.$$

We notice that $e^{itH_N^0}\mathcal{W}(\xi)e^{-itH_N^0} = \mathcal{W}(e^{itA}\xi)$ hence a simple computation yields for any $t \in \mathbb{R}$,

$$\mathcal{M}(e^{-itH_N^0}\, \tilde{\varrho}_N(t)\, e^{itH_N^0},\, N \in \mathbb{N}) = \{(e^{-itA})_*\nu, \nu \in \mathcal{M}(\tilde{\varrho}_N(t),\, N \in \mathbb{N})\} = \{(e^{-itA})_*(\tilde{\Phi}(t,0)_*\mu_0)\}.$$

Finally, remark that $\Phi(t,0) = e^{-itA} \circ \tilde{\Phi}(t,0)$. So the main Theorem 2.3 is now proved. □

7 Ground State Energy

In this section we give a proof of the mean field approximation of the ground state energy of trapped many-boson systems (Theorem 2.5). Such a result is already proved in a general framework in [32] using a quantum De Finetti theorem. Here the proof comes as a byproduct of general properties of Wigner measures and we presented here as an illustration to our phase-space approach [6, 7, 8, 9]. The proof is based on the key Lemma 7.1 below.

Lemma 7.1. *Assume* (**A1**)-(**A2**) *and suppose that A has compact resolvent. Let $\{|\Psi^{(N)}\rangle\langle\Psi^{(N)}|\}_{N\in\mathbb{N}}$ a sequence of normal states on $\vee^N \mathcal{Z}_0$ satisfying:*

$$\exists C > 0, \forall N \in \mathbb{N}, \langle \Psi^{(N)}, H_N^0 \Psi^{(N)}\rangle \leq CN. \tag{57}$$

Then any Wigner measure μ of $\{|\Psi^{(N)}\rangle\langle\Psi^{(N)}|\}_{N\in\mathbb{N}}$ satisfies the equality

$$\mu(S^1_{\mathcal{Z}_0}) = 1,$$

where $S^1_{\mathcal{Z}_0}$ is the unit sphere of the Hilbert space $\mathcal{Z}_0$.

Proof. Without loss of generality we can assume that $\mathcal{M}(|\Psi^{(N)}\rangle\langle\Psi^{(N)}|, \mathbb{N} \in N) = \{\mu\}$. Remark that the Wigner measure μ is supported on the unit ball $B(\mathcal{Z}_0)$ owing to Proposition A.6. We shall prove

$$\int_{\mathcal{Z}_0} \|z\|^2_{\mathcal{Z}_0} \, d\mu(z) \geq 1. \tag{58}$$

Indeed if (58) holds then

$$\int_{\mathcal{Z}_0} 1 - \|z\|^2_{\mathcal{Z}_0} \, d\mu(z) = 0 = \int_{B(\mathcal{Z}_0)} \underbrace{1 - \|z\|^2_{\mathcal{Z}_0}}_{\geq 0} \, d\mu(z), \text{ and } \mu(S^1_{\mathcal{Z}_0}) = 1.$$

Since A has compact resolvent then $A = \sum_{i=0}^{\infty} \lambda_i |e_i\rangle\langle e_i|$, with $(e_i)_{i\geq 0}$ is a O.N.B of $\mathcal{Z}_0$, $\lambda_i \geq 0$ and $\lim\limits_{i\to+\infty} \lambda_i = +\infty$. Hence if $C(R) := \inf\limits_{i\geq R} \lambda_i$ then

$$\lim_{R\to+\infty} C(R) = +\infty.$$

Therefore the following estimate holds true

$$\begin{aligned}
\langle \Psi^{(N)}, \sum_{i=1}^{R} |e_i\rangle\langle e_i| \, \Psi^{(N)}\rangle &= 1 - \langle \Psi^{(N)}, \sum_{i=R+1}^{\infty} |e_i\rangle\langle e_i| \, \Psi^{(N)}\rangle = 1 - \langle \Psi^{(N)}, \sum_{i=R+1}^{\infty} \frac{\lambda_i}{C(R)} |e_i\rangle\langle e_i| \Psi^{(N)}\rangle \\
&= 1 - \frac{1}{C(R)} \langle \Psi^{(N)}, A_1 \Psi^{(N)}\rangle \geq 1 - \frac{C}{C(R)},
\end{aligned}$$

since $\langle \Psi^{(N)}, A_1 \Psi^{(N)}\rangle \leq C$ by (57). Taking the limit $N \to \infty$, we get by Proposition A.5 in Appendix A

$$\lim_{N\to\infty} \langle \Psi^{(N)}, b^{Wick} \, \Psi^{(N)}\rangle = \int_{\mathcal{Z}_0} \langle z, \sum_{i=1}^{R} |e_i\rangle\langle e_i| \, z\rangle \, d\mu(z) \geq 1 - \frac{C_1}{C(R)}, \tag{59}$$

where $b(z) = \langle z, \sum_{i=1}^{R} |e_i\rangle\langle e_i| \, z\rangle \in \mathcal{P}_{1,1}(\mathcal{Z}_0)$ and $\tilde{b} = \sum_{i=1}^{R} |e_i\rangle\langle e_i| \in \mathcal{L}^\infty(\mathcal{Z}_0)$. So, we finish the proof by the dominated convergence theorem. $\square$

7.1 Upper bound

For any $\varphi \in Q(A)$, $\|\varphi\|_{\mathcal{Z}_0} = 1$, take $\Psi^{(N)} = \varphi^{\otimes N} \in Q(H_N^0)$. Compute

$$\begin{aligned}
\langle \Psi^{(N)}, H_N \, \Psi^{(N)}\rangle &= \langle \varphi^{\otimes N}, H_N^0 \varphi^{\otimes N}\rangle + \frac{N(N-1)}{2N} q_{1,2}(\varphi^{\otimes N}, \varphi^{\otimes N}) \\
&= N\langle \varphi, A\varphi\rangle + \frac{N(N-1)}{2N} q(\varphi^{\otimes 2}, \varphi^{\otimes 2}).
\end{aligned}$$

Hence

$$\frac{E(N)}{N} \leq \langle \varphi, A\varphi\rangle + \frac{1}{2} q(\varphi^{\otimes 2}, \varphi^{\otimes 2}) = h(\varphi)\,.$$

Then

$$\liminf_{N\to\infty} \frac{E(N)}{N} \leq \inf_{\varphi\in Q(A), \|\varphi\|_{\mathcal{Z}_0}=1} h(\varphi).$$

7.2 Lower bound

Let $\{\Psi^{(N)}\}_{N\in\mathbb{N}}$ be a minimizing sequence such that $\Psi^{(N)} \in Q(H_N^0)$, $\|\Psi^{(N)}\|_{\vee^N \mathcal{Z}_0} = 1$ and

$$\frac{1}{N} \langle \Psi^{(N)}, H_N \, \Psi^{(N)}\rangle \leq \frac{E(N)}{N} + \frac{1}{N}.$$

Owing to Assumption (**A2**), there exists $C_1 > 0$ such that

$$\frac{1}{N} \langle \Psi^{(N)}, H_N \, \Psi^{(N)}\rangle + C_1 \geq 0,$$

and equivalently

$$\frac{1}{N}\langle \Psi^{(N)}, H_N\, \Psi^{(N)}\rangle + C_1 \geq \langle \Psi^{(N)}, b(z)^{Wick}_{|\varepsilon=\frac{1}{N}}\, \Psi^{(N)}\rangle \geq 0\,,$$

where b is a non-negative monomial on $Q(A)$ given by

$$b(z) = \frac{1}{2}\langle z^{\otimes 2}, A\otimes 1 + 1\otimes A\, z^{\otimes 2}\rangle + \frac{1}{2}q(z^{\otimes 2}, z^{\otimes 2}) + C_1\langle z^{\otimes 2}, z^{\otimes 2}\rangle \in \mathcal{Q}_{2,2}(A)\,.$$

Wick quantization and the classes of symbols $\mathcal{Q}_{p,q}(A)$ are introduced in Appendix A. Now Proposition A.7 yields

$$\liminf_{N\to\infty} \frac{E(N)}{N} + C_1 \geq \int_{Q(A)} b(z)\, d\mu(z) + C_1 = \int_{Q(A)} h(z)\, d\mu(z) + C_1,$$

where μ is any Wigner measure of $\{|\tilde{\Psi}^{(N)}\rangle\langle\tilde{\Psi}^{(N)}|\}_{N\in\mathbb{N}}$. So using Lemma 7.1 we obtain the desired lower bound.

A The Wick quantization and Wigner measures

Although the mean-field problem considered in this paper deals with many-body Schrödinger Hamiltonians of the form of H_N given in (11), it is conceptually important to see H_N as a second quantization of the classical energy (17). This in particular allows to understand the phase-space analysis hidden in the mean-field approximation and provides convenient tools to analyze phase-space distributions of states as well as their evolutions.

A.1 Wick quantization

So, for reader's convenience we briefly recall in this appendix the ε-dependent Wick quantization in the Fock spaces and provide some general properties. Let $\mathcal{Z}_0$ be a complex Hilbert space and consider the symmetric Fock space

$$\Gamma_s(\mathcal{Z}_0) = \bigoplus_{n=0}^{\infty} \mathcal{S}_n(\mathcal{Z}_0^{\otimes n}) = \bigoplus_{n=0}^{\infty} \vee^n \mathcal{Z}_0\,,$$

where $\mathcal{S}_n$ denotes the symmetrization operator given by (7). On this Fock space there exists a realization of the following ε-dependant canonical commutation relations (CCR):

$$\big[a(z_1), a^*(z_2)\big] = \varepsilon\langle z_1\,,\, z_2\rangle\ \mathrm{Id}, \qquad \big[a^*(z_1), a^*(z_2)\big] = \big[a(z_1), a(z_2)\big] = 0\,,\ \ \varepsilon > 0\,,$$

given by the ε-dependant annihilation and creation operators,

$$\begin{aligned} a(z_1)_{|\vee^N\mathcal{Z}_0} &= \sqrt{\varepsilon N}\langle z_1| \otimes \mathrm{Id}_{|\vee^{N-1}\mathcal{Z}_0}, \\ a^*(z_2)_{|\vee^N\mathcal{Z}_0} &= \sqrt{\varepsilon(N+1)}\,\mathcal{S}_{N+1}(|z_2\rangle \otimes \mathrm{Id}_{|\vee^N\mathcal{Z}_0})\,. \end{aligned}$$

The Weyl operator is also ε-dependent and it is defined for any $\xi \in \mathcal{Z}_0$ by the formula:

$$\mathcal{W}(\xi) = e^{i\frac{a(\xi)+a^*(\xi)}{\sqrt{2}}}\,. \tag{60}$$

For $i = 1, \cdots, n$ and C an operator on $\mathcal{Z}_0$, we denote

$$C_i = 1^{\otimes(i-1)} \otimes C \otimes 1^{\otimes(n-i)},$$

where the operator C in the right hand side acts on the i^{th} component. The second quantization $d\Gamma(C)$ is the ε-dependent operator defined by

$$d\Gamma(C)_{|\vee^n\mathcal{Z}_0} = \varepsilon\sum_{i=0}^{n} C_i\,.$$

In particular, the ε-dependent number operator is

$$\mathbf{N} := d\Gamma(\mathrm{Id})\,. \tag{61}$$

The Wick quantization is a map that corresponds to a monomial $z \in \mathcal{Z}_0 \mapsto b(z)$ an operator on the Fock space (the function $b(z)$ is called a symbol in connection with the pseudo-differential calculus). It is related to the normal ordering of products of creation-annihilation operators which is a well treated subject in standard textbooks (see for instance [13, 17]). Here we follow the presentation in [6] which stresses the symbol-operator correspondence and which is more convenient for our purpose. We introduce below two type of classes of symbols $\mathcal{P}_{p,q}(\mathcal{Z}_0)$ and $\mathcal{Q}_{p,q}(A)$ and stress their main properties.

For all $p, q \in \mathbb{N}$, we denote $\mathcal{P}_{p,q}(\mathcal{Z}_0)$, resp. $\mathcal{P}^{\infty}_{p,q}(\mathcal{Z}_0)$, the space of complex-valued monomials on $\mathcal{Z}_0$, defined according to the conditions:

$$\begin{aligned} b \in \mathcal{P}_{p,q}(\mathcal{Z}_0) &\Leftrightarrow \exists!\, \tilde{b} \in \mathcal{L}(\vee^p \mathcal{Z}_0, \vee^q \mathcal{Z}_0),\ b(z) = \langle z^{\otimes q}, \tilde{b} z^{\otimes p}\rangle\,, \\ b \in \mathcal{P}^{\infty}_{p,q}(\mathcal{Z}_0) &\Leftrightarrow \exists!\, \tilde{b} \in \mathcal{L}^{\infty}(\vee^p \mathcal{Z}_0, \vee^q \mathcal{Z}_0),\ b(z) = \langle z^{\otimes q}, \tilde{b} z^{\otimes p}\rangle. \end{aligned} \tag{62}$$

Here $\mathcal{L}$ and $\mathcal{L}^{\infty}$ refer to the space of bounded operators and the space of compact operators respectively.

Definition A.1. *For $\varepsilon > 0$ and for each symbol $b \in \mathcal{P}_{p,q}(\mathcal{Z}_0)$, with $\tilde{b}$ as in (62), we associate an operator b^{Wick}: $\bigoplus_{n\geq 0}^{alg} \vee^n \mathcal{Z}_0 \longrightarrow \bigoplus_{n\geq 0}^{alg} \vee^n \mathcal{Z}_0$, given by*

$$b^{Wick}_{|\vee^n \mathcal{Z}_0} = 1_{[p,+\infty)}(n) \frac{\sqrt{n!(n+q-p)!}}{(n-p)!} \varepsilon^{\frac{p+q}{2}} \mathcal{S}_{n-p+q}(\tilde{b} \otimes \mathrm{Id}^{\otimes(n-p)}) \in \mathcal{L}(\vee^n \mathcal{Z}_0, \vee^{n+q-p} \mathcal{Z}_0)\,. \tag{63}$$

The Wick quantization map depends in the parameter $\varepsilon > 0$, however for simplicity we omit this dependence in the notation of b^{Wick}. By linearity one can extend this quantization to any finite sum in $\oplus^{alg}_{p,q\geq 0} \mathcal{P}_{p,q}(\mathcal{Z}_0)$. Remark however that the classical energy functional $h(z) = \langle z, Az\rangle + \frac{1}{2} q(z^{\otimes 2}, z^{\otimes 2})$ (given in (17)) is not in $\oplus^{alg}_{p,q\geq 0} \mathcal{P}_{p,q}(\mathcal{Z}_0)$ unless A and q are bounded. So, in order to extend the above quantization procedure to more interesting symbols, we introduce in the sequel another class of monomials $\mathcal{Q}_{p,q}(A)$.

Let A be a given non-negative self-adjoint operator on $\mathcal{Z}_0$. Let H^0_n denotes, for each $n \in \mathbb{N}$, the operator on $\vee^n \mathcal{Z}_0$ defined according to (8), i.e.:

$$H^0_{n_{|\vee^n \mathcal{Z}_0}} = \sum_{i=1}^{n} A_i\,.$$

For simplicity we denote

$$\mathfrak{Q}_n := Q(H^0_n) \subset \vee^n \mathcal{Z}_0 \quad \text{and} \quad Q_n := Q(\sum_{i=1}^{n} A_i) \subset \otimes^n \mathcal{Z}_0\,,$$

with Q_n is a subspace possessing non symmetric vectors satisfying $\mathfrak{Q}_n \subset Q_n$, $\mathcal{S}_n Q_n = \mathfrak{Q}_n$ and $Q_n, \mathfrak{Q}_n$ are respectively dense in $\otimes^n \mathcal{Z}_0, \vee^n \mathcal{Z}_0$. Remember that Q_n and $\mathfrak{Q}_n$ are Hilbert spaces when they are equipped with the graph norm

$$\|u\|_{Q_n} = \|u\|_{\mathfrak{Q}_n} = \sqrt{\langle u, \sum_{i=1}^{n} [A_i + 1]\, u\rangle}\,, \quad \forall u \in Q_n. \tag{64}$$

We denote by Q'_n and $\mathfrak{Q}'_n$ respectively the dual spaces of Q_n and $\mathfrak{Q}_n$ with respect to the scalar product of $\otimes^n \mathcal{Z}_0$.
For all $p, q \in \mathbb{N}$, we define the class of symbols $\mathcal{Q}_{p,q}(A)$ as the space of complex-valued monomials on $Q(A)$ verifying

$$b \in \mathcal{Q}_{p,q}(A) \Leftrightarrow \exists!\, \tilde{b} \in \mathcal{L}(\mathfrak{Q}_p, \mathfrak{Q}'_q),\ \forall z \in Q(A),\ b(z) = \langle z^{\otimes q}, \tilde{b} z^{\otimes p}\rangle_{\otimes^q \mathcal{Z}_0}\,. \tag{65}$$

Let $b \in \mathcal{Q}_{p,q}(A)$ and $\tilde{b}$ as in (65), then the map defined for any $\varphi_1, \cdots, \varphi_n \in Q(A)$ by

$$\tilde{b} \otimes 1^{(n-p)}\ \mathcal{S}_p \otimes 1^{(n-p)} \varphi_1 \otimes \cdots \otimes \varphi_n = \Big(\tilde{b} \mathcal{S}_p\left(\varphi_1 \otimes \cdots \varphi_p\right)\Big) \otimes \varphi_{p+1} \cdots \otimes \varphi_n\,, \tag{66}$$

extends by linearity and continuity to a bounded operator from Q_n into Q'_{n-p+q} since for any $\Phi^{(n)} \in \otimes^{alg,n} Q(A)$

$$\begin{aligned} \|\tilde{b} \otimes 1^{(n-p)}\ \mathcal{S}_p \otimes 1^{(n-p)}\, \Phi^{(n)}\|_{Q'_{n-p+q}} &= \|(\sum_{i=1}^{n-p+q} A_i + 1)^{-\frac{1}{2}} \mathcal{S}_q \tilde{b} \mathcal{S}_p (\sum_{i=1}^{p} A_i + 1)^{-\frac{1}{2}} (\sum_{i=1}^{p} A_i + 1)^{\frac{1}{2}} \Phi^{(n)}\| \\ &\leq \|\tilde{b}\|_{\mathcal{L}(\mathfrak{Q}_p, \mathfrak{Q}'_q)} \|\Phi^{(n)}\|_{Q_n}\,, \end{aligned}$$

and the subspace $\otimes^{alg,n}Q(A)$ is a form core for $\sum_{i=1}^{n} A_i$. As a consequence, we see that

$$\mathcal{S}_{n-p+q}\,\tilde{b}\otimes 1^{(n-p)}\,\mathcal{S}_n = \mathcal{S}_{n-p+q}\,\tilde{b}\otimes 1^{(n-p)}\,\mathcal{S}_p\otimes 1^{(n-p)}\,\mathcal{S}_n \in \mathcal{L}(\mathfrak{Q}_n, \mathfrak{Q}'_{n-p+q})\,.$$

Definition A.2. *For each symbol $b \in \mathcal{Q}_{p,q}(A)$, with $\tilde{b}$ as in (65), we associate an operator b^{Wick}: $\bigoplus_{n\geq 0}^{alg}\mathfrak{Q}_n \longrightarrow \bigoplus_{n\geq 0}^{alg}\mathfrak{Q}'_n$, given by*

$$b^{Wick}_{|\mathfrak{Q}_n} = 1_{[p,+\infty)}(n)\frac{\sqrt{n!(n+q-p)!}}{(n-p)!}\varepsilon^{\frac{p+q}{2}}\mathcal{S}_{n-p+q}(\tilde{b}\otimes 1^{\otimes(n-p)}) \in \mathcal{L}(\mathfrak{Q}_n, \mathfrak{Q}'_{n-p+q})\,. \qquad (67)$$

Actually $b^{Wick}_{|\mathfrak{Q}_n}$ can also be understood as a bounded sesquilinear form on $\mathfrak{Q}_n \times \mathfrak{Q}_{n-p+q}$. Remark that we have always the inclusion $\mathcal{P}_{p,q}(\mathcal{Z}_0) \subset \mathcal{Q}_{p,q}(A)$. Furthermore, the class $\mathcal{Q}_{p,q}(A)$ depends on the operator A and if A is bounded on $\mathcal{Z}_0$ then $\mathcal{Q}_{p,q}(A)$ coincides with $\mathcal{P}_{p,q}(\mathcal{Z}_0)$.

Examples: Let q be a quadratic form on Q_2 satisfying the assumption (**A2**) and $\tilde{q}$ defined according to (9). The main examples of interest here are

$$b_0(z) = \langle z, Az\rangle \in \mathcal{Q}_{1,1}(A) \qquad \text{with} \quad \tilde{b}_0 = A\,,$$
$$b(z) = q(z^{\otimes 2}, z^{\otimes 2}) \in \mathcal{Q}_{2,2}(A) \qquad \text{with} \quad \tilde{b} = \mathcal{S}_2\tilde{q}\mathcal{S}_2\,,$$

and

$$h(z) = \langle z, Az\rangle + \frac{1}{2}q(z^{\otimes 2}, z^{\otimes 2}) \in \mathcal{Q}_{1,1}(A) + \mathcal{Q}_{2,2}(A)\,. \qquad (68)$$

So using the Wick quantization given in Definition A.2, one obtains the following equality in the sense of quadratic forms for any $\Psi^{(N)}, \Phi^{(N)} \in \mathfrak{Q}_N$,

$$\langle \Psi^{(N)}, H_N\Phi^{(N)}\rangle = \langle \Psi^{(N)}, \varepsilon^{-1}h^{Wick}\Phi^{(N)}\rangle\,, \quad \text{when} \quad \varepsilon = \frac{1}{N}\,.$$

This identity shows the relationship between the Hamiltonian of many-boson systems in the mean-field scaling and the Wick quantization of symbols in $\mathcal{Q}_{p,q}(A)$ with the semiclassical parameter ε. In fact most of the information we need in the analysis of the mean field approximation comes from general properties of the classes $\mathcal{Q}_{p,q}(A)$ stated in Proposition A.3 below.

The linear space $\mathcal{Q}_{p,q}(A)$ is a subset of the space of continuous functions on $Q(A)$ and can be equipped with a convenient convergence topology. We say that a sequence $(c_m)_{m\in N}$ in $\mathcal{Q}_{p,q}(A)$ is b-convergent to a function $c(z)$ iff

$$c_m \xrightarrow{b} c \Leftrightarrow \forall z \in Q(A),\ c_m(z) \to c(z) \text{ and } (||\tilde{c}_m||_{\mathcal{L}(\mathfrak{Q}_p,\mathfrak{Q}'_q)})_{m\in\mathbb{N}} \text{ is bounded }\,.$$

Proposition A.3. *For any $b \in \mathcal{Q}_{p,q}(A)$ and $(c_m)_{m\in N}$ a sequence in $\mathcal{Q}_{p,q}(A)$, we have:*
(i) $\bar{b} \in \mathcal{Q}_{q,p}(A)$ and

$$\left(b^{Wick}_{|\mathfrak{Q}_n}\right)^* = \bar{b}^{Wick}_{|\mathfrak{Q}_{n-p+q}}\,.$$

(ii) For any $t \in \mathbb{R}$, $b_t(z) := b(e^{-itA}z) \in \mathcal{Q}_{p,q}(A)$ with

$$e^{i\frac{t}{\varepsilon}d\Gamma(A)}b^{Wick}e^{-i\frac{t}{\varepsilon}d\Gamma(A)} = b_t^{Wick}\,.$$

(iii) There exists a constant $C_{p,q} > 0$ such that for any $\Psi^{(n)} \in \mathfrak{Q}_n$, $\Phi^{(m)} \in \mathfrak{Q}_m$ with $m = n-p+q$ and $\varepsilon = \frac{1}{n}$,

$$\left|\langle \Phi^{(m)}, b^{Wick}\,\Psi^{(n)}\rangle\right| \leq C_{p,q}\,\left\|\tilde{b}\right\|_{\mathcal{L}(\mathfrak{Q}_n,\mathfrak{Q}'_m)}\,\left\|(A_1+1)^{\frac{1}{2}}\Phi^{(m)}\right\|\,\left\|(A_1+1)^{\frac{1}{2}}\Psi^{(n)}\right\|\,.$$

(iv) If $c_m \xrightarrow{b} c$ then $c \in \mathcal{Q}_{p,q}(A)$ and c_m^{Wick} converges weakly to c^{Wick} in $\mathcal{L}(\mathfrak{Q}_n, \mathfrak{Q}'_{n-p+q})$.
(v) For any $\xi \in Q(A)$ the symbol $b(\cdot+\xi)$ belongs to $\oplus_{p,q\in\mathbb{N}}^{alg}\mathcal{Q}_{p,q}(A)$ and the identity

$$b^{Wick}\mathcal{W}(\frac{\sqrt{2}}{i\varepsilon}\xi) = \mathcal{W}(\frac{\sqrt{2}}{i\varepsilon}\xi)b(z+\xi)^{Wick}\,, \qquad (69)$$

holds in the sense of sesquilinear forms on $\mathfrak{Q}_{n_1} \times \mathfrak{Q}_{n_2}$ for any $n_1, n_2 \in \mathbb{N}$.

Proof. (i) According to (65), we have

$$\bar{b}(z) = \overline{b(z)} = \langle \tilde{b}\, z^{\otimes q}, z^{\otimes p}\rangle = \langle z^{\otimes q}, \tilde{b}^*\, z^{\otimes p}\rangle\,,$$

where $\tilde{b}^* \in \mathcal{L}(\mathfrak{Q}_q, \mathfrak{Q}'_p)$ is the adjoint of $\tilde{b} \in \mathcal{L}(\mathfrak{Q}_p, \mathfrak{Q}'_q)$. Let $\Phi^{(n)} \in \vee^{alg,n}Q(A)$, $\Psi^{(m)} \in \vee^{alg,m}Q(A)$ with $m = n - p + q$, then we have

$$\begin{aligned}
\langle b^{Wick}\Phi^{(n)}, \Psi^{(m)}\rangle &= 1_{[p,+\infty)}(n)\frac{\sqrt{n!m!}}{(n-p)!}\varepsilon^{\frac{p+q}{2}}\ \langle \tilde{b}\otimes 1^{\otimes(n-p)}\,\Phi^{(n)}, \Psi^{(m)}\rangle \\
&= 1_{[p,+\infty)}(n)\frac{\sqrt{n!m!}}{(n-p)!}\varepsilon^{\frac{p+q}{2}}\ \langle \Phi^{(n)}, \tilde{b}^*\otimes 1^{\otimes(n-p)}\Psi^{(m)}\rangle \\
&= \langle \Phi^{(n)}, \bar{b}^{Wick}\Psi^{(m)}\rangle\,.
\end{aligned}$$

Since $\vee^{alg,n}Q(A)$ is dense in the Hilbert space $(\mathfrak{Q}_n, ||.||_{\mathfrak{Q}_n})$, the above identity extends to any $\Phi^{(n)} \in \mathfrak{Q}_n$ and $\Psi^{(m)} \in \mathfrak{Q}_m$.

(ii) For any $t \in \mathbb{R}$ and $n \in \mathbb{N}$ the operator $(e^{-itA})^{\otimes n} : \mathfrak{Q}_n \to \mathfrak{Q}_n$ is bounded and extends by duality to a bounded operator on $\mathfrak{Q}'_n$. Hence for any $z \in Q(A)$,

$$b_t(z) := \langle (e^{-itA}z)^{\otimes q}, \tilde{b}\,(e^{-itA}z)^{\otimes p}\rangle = \langle z^{\otimes q}, (e^{-itA})^{\otimes q}\,\tilde{b}\,(e^{-itA})^{\otimes p}\ z^{\otimes p}\rangle\,,$$

and $\tilde{b}_t = (e^{-itA})^{\otimes q}\,\tilde{b}\,(e^{-itA})^{\otimes p}$ belongs to $\mathcal{L}(\mathfrak{Q}_p, \mathfrak{Q}'_q)$. Let $\Phi^{(n)} \in \mathfrak{Q}_n$, $\Psi^{(m)} \in \mathfrak{Q}_m$ with $m = n - p + q$, then we have

$$\begin{aligned}
\langle \Psi^{(m)}, e^{i\frac{t}{\varepsilon}d\Gamma(A)}b^{Wick}e^{-i\frac{t}{\varepsilon}d\Gamma(A)}\Phi^{(n)}\rangle &= 1_{[p,+\infty)}(n)\frac{\sqrt{n!m!}}{(n-p)!}\varepsilon^{\frac{p+q}{2}}\ \langle \Psi^{(m)}, (e^{-itA})^{\otimes m}\ \tilde{b}\otimes 1^{\otimes(n-p)}\,(e^{-itA})^{\otimes n}\Phi^{(n)}\rangle \\
&= 1_{[p,+\infty)}(n)\frac{\sqrt{n!m!}}{(n-p)!}\varepsilon^{\frac{p+q}{2}}\ \langle \Psi^{(m)}, \tilde{b}_t\otimes 1^{\otimes(n-p)}\Phi^{(n)}\rangle \\
&= \langle \Psi^{(m)}, b_t^{Wick}\Phi^{(n)}\rangle\,.
\end{aligned}$$

(iii) A simple estimate gives

$$\begin{aligned}
\left|\langle \Phi^{(m)}, b^{Wick}\,\Psi^{(n)}\rangle\right| &\leq \frac{\sqrt{n!(n+q-p)!}}{(n-p)!}\varepsilon^{\frac{p+q}{2}}\ \Big|\Big\langle (H_q^0+1)^{\frac12}\otimes 1^{\otimes(m-q)}\Phi^{(m)}; \\
&\qquad \left((H_q^0+1)^{-\frac12}\tilde{b}(H_p^0+1)^{-\frac12}\right)\otimes 1^{\otimes(n-p)}\,(H_p^0+1)^{\frac12}\otimes 1^{\otimes(n-p)}\Psi^{(n)}\Big\rangle\Big| \\
&\leq \left\|\tilde{b}\right\|_{\mathcal{L}(\mathfrak{Q}_n,\mathfrak{Q}'_m)}\ \left\|(H_q^0+1)^{\frac12}\otimes 1^{\otimes(m-q)}\Phi^{(m)}\right\|\ \left\|(H_p^0+1)^{\frac12}\otimes 1^{\otimes(n-p)}\Psi^{(n)}\right\|\,.
\end{aligned}$$

Using the symmetry of the vectors $\Phi^{(m)}$ (resp. $\Psi^{(n)}$), we remark

$$\left\|(H_q^0+1)^{\frac12}\otimes 1^{\otimes(m-q)}\Phi^{(m)}\right\|^2 = \langle \Phi^{(m)}, (\sum_{i=1}^{q} A_i + 1)\,\Phi^{(m)}\rangle = \langle \Phi^{(m)}, (qA_1 + 1)\,\Phi^{(m)}\rangle\,.$$

(iv) Thanks to a polarization formula the monomial c_m determines uniquely the operator $\tilde{c}_m \in \mathcal{L}(\mathfrak{Q}_p, \mathfrak{Q}'_q)$. In fact for any $\Phi^{(q)} \in \vee^{alg,q}Q(A)$ and $\Psi^{(p)} \in \vee^{alg,q}Q(A)$ the quantity $\langle \Phi^{(q)}, \tilde{c}_m\,\Psi^{(p)}\rangle$ can be written as a linear combination of $(c_m(z_i))_{i\in I}$ where I is a finite set and z_i are given points in $Q(A)$. Therefore, for any $\Phi^{(q)} \in \vee^{alg,q}Q(A)$ and $\Psi^{(p)} \in \vee^{alg,p}Q(A)$ the sequence $(\langle \Phi^{(q)}, \tilde{c}_m\,\Psi^{(p)}\rangle)_{m\in\mathbb{N}}$ is convergent. Since $(||\tilde{c}_m||_{\mathcal{L}(\mathfrak{Q}_p,\mathfrak{Q}'_q)})_{m\in\mathbb{N}}$ is bounded, one can prove by an $\eta/3$-argument that $\tilde{c}_m$ converges weakly to an operator $\tilde{c} \in \mathcal{L}(\mathfrak{Q}_p, \mathfrak{Q}'_q)$, i.e.:

$$\langle \Phi^{(q)}, \tilde{c}_m\,\Psi^{(p)}\rangle \underset{m\to\infty}{\to} \langle \Phi^{(q)}, \tilde{c}\,\Psi^{(p)}\rangle\,, \quad \forall\Phi^{(q)} \in \mathfrak{Q}_q, \forall\Psi^{(p)} \in \mathfrak{Q}_p\,. \tag{70}$$

Hence, $c(z) = \langle z^{\otimes q}, \tilde{c}z^{\otimes p}\rangle$ and belongs to $\mathcal{Q}_{p,q}(A)$. As a consequence of (70), the operator $\tilde{c}_m \otimes 1^{(n-p)}$ converges also weakly to $\tilde{c}\otimes 1^{(n-p)}$ in $\mathcal{L}(\mathfrak{Q}_n, \mathfrak{Q}'_{n-p+q})$ and the convergence of c_m^{Wick} towards c^{Wick} follows.

(v) The relation (69) is already proved in [6, Proposition 2.10] for symbols $b \in \mathcal{P}_{p,q}(\mathcal{Z}_0)$. In order to extend it to the class $\mathcal{Q}_{p,q}(A)$ it is enough to use the approximation argument provided by (iv). Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\chi(x) = 1$ if $\|x\| \leq 1$, $\chi(x) = 0$ if $\|x\| \geq 2$ and $0 \leq \chi \leq 1$. We denote for $m \in \mathbb{N}$, $\chi_m(x) = \chi(\frac{x}{m})$. Let $b \in \mathcal{Q}_{p,q}(A)$ and consider the sequence of symbols

$$c_m(z) = \langle z^{\otimes q}, \chi_m(H_q^0)\,\tilde{b}\,\chi_m(H_p^0)\,z^{\otimes p}\rangle \in \mathcal{P}_{p,q}(\mathcal{Z}_0) \subset \mathcal{Q}_{p,q}(A)\,.$$

The use of [6, Proposition 2.10] yields for any $\Phi^{(n_1)} \in \mathfrak{Q}_{n_1}$ and $\Psi^{(n_2)} \in \mathfrak{Q}_{n_2}$,

$$\langle \Phi^{(n_1)}, c_m^{Wick} \mathcal{W}(\frac{\sqrt{2}}{i\varepsilon}\xi)\Psi^{(n_2)}\rangle = \langle \Phi^{(n_1)}, \mathcal{W}(\frac{\sqrt{2}}{i\varepsilon}\xi)\, c_m(z+\xi)^{Wick}\, \Psi^{(n_2)}\rangle\,. \tag{71}$$

Now, it is easy to check that

$$c_m \xrightarrow{b} b\,, \quad \text{in } \mathcal{Q}_{p,q}(A).$$

Moreover $c_m(\cdot+\xi) \in \oplus^{alg}_{k,l\geq 0}\mathcal{P}_{k,l}(\mathcal{Z}_0)$,

$$\begin{aligned} c_m(z+\xi) &= \langle (z+\xi)^{\otimes q}, \tilde{c}_m\,(z+\xi)^{\otimes p}\rangle \\ &= \sum_{\substack{0\leq i\leq q\\ 0\leq j\leq p}} C_q^i C_p^j\, \langle z^{\otimes(q-i)}\otimes \xi^{\otimes i}, \mathcal{S}_q\, \tilde{c}_m\, \mathcal{S}_p\, z^{\otimes(p-j)}\otimes \xi^{\otimes j}\rangle \\ &=: \sum_{\substack{0\leq i\leq q\\ 0\leq j\leq p}} C_q^i C_p^j\, c_m^{(i,j)}(z)\,. \end{aligned}$$

So, it is clear that each monomial $c_m^{(i,j)}$ in the above sum b-converges to $b^{(i,j)} = \langle z^{\otimes(q-i)}\otimes\xi^{\otimes i}, \mathcal{S}_q\, \tilde{b}\, \mathcal{S}_p\, z^{\otimes(p-j)}\otimes \xi^{\otimes j}\rangle$ since $\tilde{c}_m$ converges weakly to $\tilde{b}$ in $\mathcal{L}(\mathfrak{Q}_p, \mathfrak{Q}_q')$. Remark also that Proposition A.4 shows for any $r\in\mathbb{N}$ that the r^{th} components of the following coherent vectors satisfy

$$\big[\mathcal{W}(\frac{\sqrt{2}}{i\varepsilon}\xi)\Psi^{(n_2)}\big]^{(r)} \in \mathfrak{Q}_r \quad \text{and} \quad \big[\mathcal{W}(\frac{\sqrt{2}}{i\varepsilon}\xi)^*\Phi^{(n_1)}\big]^{(r)} \in \mathfrak{Q}_r\,.$$

Therefore using (iv) and taking the limit $m\to\infty$ in (71) proves the claimed identity. □

A regularity property of Weyl operators: It is convenient to recall the following regularity property for the Weyl operators. Remember that the operator $d\Gamma(A)+\mathbf{N}$ is non-negative and self-adjoint on the symmetric Fock space satisfying

$$d\Gamma(A) + \mathbf{N}_{|\vee^N \mathcal{Z}_0} = H_N^0 + 1\,, \quad \text{when } \varepsilon = \frac{1}{N}\,.$$

Moreover, $d\Gamma(A)+\mathbf{N}$ has an invariant form domain with respect to the Weyl operator $\mathcal{W}(\xi)$ when $\xi\in Q(A)$. This propriety can be proved using the Faris-Lavine argument [21] and it is proved for instance in [4].

Proposition A.4. *For any $\xi\in Q(A)$ the form domain $Q(d\Gamma(A)+\mathbf{N})$ is invariant with respect to the Weyl operator $\mathcal{W}(\xi)$. Moreover, there exists uniformly in $\varepsilon\in(0,\bar{\varepsilon})$ a constant $C := C(\xi)>0$ such that*

$$\|(d\Gamma(A)+\mathbf{N})^{\frac{1}{2}}\mathcal{W}(\xi)(d\Gamma(A)+\mathbf{N}+1)^{-\frac{1}{2}}\|_{\Gamma_s(\mathcal{Z}_0)} \leq C\,, \tag{72}$$

and in particular for any $\Psi^{(N)}\in Q(H_N^0)$, $\varepsilon=\frac{1}{N}$,

$$\left\|(H_{N-1}^0+1)^{1/2}[W(\xi)\Psi^{(N)}]^{(N-1)}\right\| \leq C\left\|(H_N^0+1)^{1/2}\Psi^{(N)}\right\|,$$

where $[\mathcal{W}(\xi)\Psi^{(N)}]^{(N-1)}$ denotes the $(N-1)^{th}$ component of $\mathcal{W}(\xi)\Psi^{(N)}\in\Gamma_s(\mathcal{Z}_0)$.

A.2 Relationship with Wigner measures

Wigner measures are defined through Weyl operators nevertheless it is important for the mean-field problem to draw the link with Wick quantization. Their relationship is clarified by the following Proposition proved in [6, Theorem 6.13] and [6, Corollary 6.14].

Proposition A.5. *Let $\{|\Psi^{(N)}\rangle\langle\Psi^{(N)}|\}_{N\in\mathbb{N}}$ be a sequence of normal states on $\vee^N\mathcal{Z}_0$ satisfying:*

$$\exists C>0, \forall N\in\mathbb{N},\ \langle \Psi^{(N)}, H_N^0\Psi^{(N)}\rangle \leq CN\,,$$

and

$$\mathcal{M}(|\Psi^{(N)}\rangle\langle\Psi^{(N)}|, N\in\mathbb{N}) = \{\mu\}.$$

Then, for any $b \in \oplus^{alg}_{p,q\geq 0}\mathcal{P}^{\infty}_{p,q}(\mathcal{Z}_0)$,

$$\lim_{\substack{N\to+\infty\\ \varepsilon N=1}} \langle \Psi^{(N)}, b^{Wick}\Psi^{(N)}\rangle = \int_{\mathcal{Z}_0} b(z)\, d\mu(z)\,,$$

$$\lim_{\substack{N\to+\infty\\ \varepsilon N=1}} \langle \Psi^{(N)}, \mathcal{W}(\xi)\, b^{Wick}\Psi^{(N)}\rangle = \int_{\mathcal{Z}_0} e^{i\mathrm{Re}\langle z,\xi\rangle}\, b(z)\, d\mu(z)\ ,$$

The following a priori estimate is a consequence of [9, Proposition 3.11], [9, Lemma 3.13], [8, Lemma 2.14] and [9, Lemma 3.12].

Proposition A.6. *Let* $\{|\Psi^{(N)}\rangle\langle\Psi^{(N)}|\}_{N\in\mathbb{N}}$ *a sequence of normal states on* $\vee^n\mathcal{Z}_0$ *satisfying:*

$$\exists C>0, \forall N\in\mathbb{N},\ \langle \Psi^{(N)}, H^0_N\Psi^{(N)}\rangle \leq CN\,,$$

and

$$\mathcal{M}(|\Psi^{(N)}\rangle\langle\Psi^{(N)}|, N\in\mathbb{N}) = \{\mu\}.$$

Then the Wigner measure μ *is carried by* $Q(A)$ *(i.e.:* $\mu(Q(A))=1$*) and its restriction to* $Q(A)$ *is a Borel probability measure on* $(Q(A), ||\cdot||_{Q(A)})$ *fulfilling*

$$\int_{\mathcal{Z}_0} \|z\|^2_{Q(A)}\, d\mu(z) \leq C\,,$$

$$\text{and} \qquad \mu(B(\mathcal{Z}_0)) = 1\,,$$

where $B(\mathcal{Z}_0)$ *is the unit ball of* $\mathcal{Z}_0$*.*

Some kind of a Fatou's lemma for Wigner measures holds true.

Proposition A.7. *Let* $\{|\Psi^{(N)}\rangle\langle\Psi^{(N)}|\}_{N\in\mathbb{N}}$ *be a sequence of normal states on* $\vee^N\mathcal{Z}_0$ *satisfying:*

$$\exists C>0, \forall N\in\mathbb{N},\ \langle \Psi^{(N)}, H^0_N\Psi^{(N)}\rangle \leq CN\,,$$

and

$$\mathcal{M}(|\Psi^{(N)}\rangle\langle\Psi^{(N)}|, N\in\mathbb{N}) = \{\mu\}.$$

Then for any $b\in\mathcal{Q}_{p,p}(A)$ *such that* $\tilde{b}\geq 0$,

$$\liminf_{\substack{N\to+\infty\\ \varepsilon N=1}} \langle \Psi^{(N)}, b^{Wick}\Psi^{(N)}\rangle \geq \int_{\mathcal{Z}_0} b(z)\, d\mu(z)\,.$$

Proof. Since $b\in\mathcal{Q}_{p,p}(A)$, $b(z)=\langle z^{\otimes p}, \tilde{b}\, z^{\otimes p}\rangle$ with $\tilde{b}\in\mathcal{L}(\mathfrak{Q}_p,\mathfrak{Q}'_p)$, $\tilde{b}\geq 0$, then the quadratic form

$$(\Psi,\Phi)\in Q(H^0_p)\times Q(H^0_p)\to\langle\Psi,\tilde{b}\,\Phi\rangle\,,$$

is closed and non-negative. Hence by [38, Theorem VIII] there exists a unique self-adjoint operator on $\vee^p\mathcal{Z}_0$, denoted by B, such that $\langle\Psi,\tilde{b}\,\Phi\rangle=\langle\Psi,B\,\Phi\rangle$ for any $\Psi,\Phi\in D(B)$ and $D(B)$ is dense in $Q(H^0_p)$. Moreover, the inequality $0\leq B\leq c\,H^0_p$ holds in the sense of quadratic forms on $Q(H^0_p)\subset Q(B)$. So, when $\varepsilon=\frac{1}{N}$,

$$\langle \Psi^{(N)}, b^{Wick}\Psi^{(N)}\rangle = \frac{N!}{N^p(N-p)!}\langle\Psi^{(N)}, B\otimes 1^{(N-p)}\Psi^{(N)}\rangle \geq \frac{N!}{N^p(N-p)!}\langle\Psi^{(N)}, \chi_m(B)B\otimes 1^{(N-p)}\Psi^{(N)}\rangle\,,$$

where χ_m is a suitable cutoff function such that $0\leq\chi_m\leq 1$ and $\chi_m\to 1$ when $m\to\infty$. For any compact operator C on $\vee^p\mathcal{Z}_0$ satisfying $0\leq C\leq\chi_m(B)B$, one get

$$\langle \Psi^{(N)}, b^{Wick}\Psi^{(N)}\rangle \geq \frac{N!}{N^p(N-p)!}\langle\Psi^{(N)}, C\otimes 1^{(N-p)}\Psi^{(N)}\rangle\,.$$

So using Proposition A.5 one obtains

$$\liminf_{\substack{N\to\infty\\ \varepsilon N=1}}\langle \Psi^{(N)}, b^{Wick}\Psi^{(N)}\rangle \geq \int_{\mathcal{Z}_0}\langle z^{\otimes p}, C\, z^{\otimes p}\rangle\, d\mu,$$

for any non-negative compact operator C such that $C\leq\chi_m(B)B$. Remark that there exists a sequence of such operators C_k which converges strongly to $\chi_m(B)B$. Therefore using Proposition A.6 and dominated convergence one obtains

$$\liminf_{\substack{N\to\infty\\ \varepsilon N=1}}\langle \Psi^{(N)}, b^{Wick}\Psi^{(N)}\rangle \geq \int_{\mathcal{Z}_0}\langle z^{\otimes p}, B\, z^{\otimes p}\rangle\, d\mu = \int_{Q(A)} b(z)\, d\mu\,.$$

□

B Measure valued solutions to continuity equation

Denote A a non-negative selfadjoint operator. Recall that $Q(A)$ is the form domain and $Q'(A)$ its dual. We consider in a weak sense the Liouville's equation on a bounded open interval $I \subset \mathbb{R}$,

$$\partial_t \mu_t + \nabla^T (v.\mu_t) = 0\,,$$

as the following integral equation,

$$\int_I \int_{Q'(A)} \partial_t \varphi(t,x) + \mathrm{Re}\langle v_t(x), \nabla \varphi(t,x)\rangle_{Q'(A)}\, d\mu_t(x)\, dt = 0, \quad \forall \varphi \in \mathcal{C}^\infty_{0,cyl}(I \times Q'(A))\,, \tag{73}$$

where μ_t belongs to $\mathfrak{P}(Q(A))$. The plan is to give an uniqueness result for the velocity field introduced earlier:

$$v_t(z) := -ie^{itA}\partial_{\bar{z}} q_0(e^{-itA}z) : \mathbb{R} \times Q(A) \to \mathcal{Z}_0,$$

where $\partial_{\bar{z}} q_0$ is defined in (12). We shall consider the local well posedness of the Cauchy problem in $Q(A)$

$$\partial_t \gamma(t) = v_t(\gamma(t)), \gamma(0) = z, z \in Q(A), \tag{74}$$

for the Borel velocity field $v_t : Q(A) \to \mathcal{Z}_0$. The following theorem provides the link between the Liouville equation (73) satisfied by the velocity field $v_t(.)$ and the Cauchy problem (74) on a infinite dimensional separable Hilbert space. For more details on this topic and for the proof of the following Theorem, we refer to [5].

Theorem B.1. *Let $v : \mathbb{R} \times Q(A) \to \mathcal{Z}_0$ be a (non-autonomous) continuous vector field satisfying Assumption* (**C1**). *Let $t \in I \to \mu_t \in \mathfrak{P}(Q(A))$ be a weakly narrowly continuous solution in $\mathfrak{P}(Q'(A))$ of the Liouville equation* (73) *defined on an open bounded interval I. Assume additionally that:*

(i) There exists $C > 0$ such that $\int_I \int_{Q(A)} ||x||^2_{Q(A)} d\mu_t(x)dt \leq C$.

(ii) There exists an open Ball B of $\mathcal{Z}_0$ such that $\mu_t(B) = 1$ for all $t \in I$.

(iii) For $s \in I$ and any $z \in Q(A) \cap B$ there exists a strong solution of (13) *defined on $\bar{I}$ with Definition in Assumption* (**C2**)*-(ii) satisfied.*

Then $\mu_t = \Phi(t,s)_\sharp \mu_s$ for all $t \in I$ with $\Phi(t,s)$ is the local flow of the initial value problem (13)*. Additionally, if the curve $t \to \mu_t$ is defined on $\mathbb{R}$ and the above assumptions still satisfied for any arbitrary bounded open interval $I \subset \mathbb{R}$, then $\mu_t = \Phi(t,s)_\sharp \mu_s$ for all $t, s \in \mathbb{R}$.*

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